An Application of Weighted Theta Functions to t-core Partitions and Numerical Semigroups

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We first present a problem about the set of positive integers represented by a certain quadratic form subject to some additional constraints. This particular problem is motivated by an application to combinatorics, which we explain below. Let

$$Q_5(x_1, x_2, x_3, x_4) = 2\left(\sum_{j=1}^{4} x_j^2\right) - \sum_{1\le j<k\le 4} x_j x_k.$$

This quadratic form is associated to a lattice, in this case a scaled copy of A_4^*. We do not consider integer inputs but (x_1, x_2, x_3, x_4) satisfying $x_j \in \mathbb{Z} + \frac{j}{5}$ for $1 \le j \le 4$. Knowing the set of integers represented by this form subject to this condition is equivalent to knowing the possible norms of vectors in a particular translate of this lattice. Standard results in the theory of quadratic forms in four or more variables imply that the vectors of norm N in this lattice translate asymptotically become equidistributed. For our particular application these types of asymptotic results are not sufficient. We want to determine exactly the set of positive integers represented by $Q_5(x_1, x_2, x_3, x_4)$ subject to the above constraint and also satisfying $\max_{j\ne k} \frac{x_j}{x_k} \le 2$. Our main result is the following.

Theorem 1 (Elkies, K.)**.** *For each $n \ge 1$, $n \notin \{1, 2, 3, 6\}$, there exist (x_1, x_2, x_3, x_4) satisfying $x_j \in \mathbb{Z} + \frac{j}{5}$, $\max_{j\ne k} \frac{x_j}{x_k} \le 2$, and $Q_5(x_1, x_2, x_3, x_4) = n$.*

This result has consequences in the study of t-core partitions and numerical semigroups. A *partition* λ of n is nonincreasing list of positive integers, $\lambda_k \ge \lambda_{k-1} \ge \cdots \ge \lambda_1 \ge 1$, that sums to n. We can represent a partition by its *Young diagram*, an array of k left-justified rows of boxes with λ_k boxes in the first row, λ_{k-1} boxes in the second row, and so on, down to λ_1 boxes in the last row. Each box in the Young diagram has a *hook length*, the number of boxes in the hook attached to this box. Hook lengths play an important role in the correspondence between irreducible representations of S_n and partitions of n. The *hook set* of a partition is the set of hook lengths of boxes of the Young diagram. A *t-core partition* is a partition none of whose hook lengths is divisible by t. We recall the t-core theorem of Granville and Ono [2].

Theorem 2 (Granville-Ono, 1996)**.** *Fix $t \ge 4$. For any $n \ge 1$, there exists a t-core partition of n.*

The proof of the t-core theorem involves studying the generating function for t-cores as a modular form and carefully considering the coefficients of its q-expansion. We will use similar modular forms methods to strengthen the Granville-Ono theorem in the case $t = 5$.

A *numerical semigroup* S is an additive submonoid of $\{0, 1, 2, \ldots, \}$ with finite complement. The size of this complement is known as the *genus* of S, denoted

$g(S)$, terminology which comes from the theory of Weierstrass semigroups on algebraic curves. The *weight* of S, denoted $w(S)$, is the sum of the elements of $\mathbb{N}\setminus S$ minus $g(S)(g(S)+1)/2$. The connection between numerical semigroups and t-core partitions is summarized in the following propositions, the second of which builds on work of Bras-Amorós and de Mier [1].

Proposition 1. *The hook set of a partition is the complement of a numerical semigroup.*

Proposition 2. *Given a numerical semigroup S there is a unique partition $\lambda(S)$ with hook set equal to $\mathbb{N} \setminus S$ and satisfying the additional property that for any integer j in the hook set there is a box in the first column of the Young diagram of λ with hook length equal to j.*
Suppose m is the smallest nonzero element of S. The $\lambda(S)$ is an m-core partition but not a t-core for any $t < m$ and the size of $\lambda(S)$ is $w(S) + g(S)$.

We can express $w(S) + g(S)$ in terms of a particular generating set of S known as the Apéry set. If S has smallest nonzero element n, then $w(S)+g(S)$ is equal to an inhomogeneous quadratic form in $m-1$ variables evaluated at inputs coming from the Apéry set of S. A simple change of variables gives the following quadratic form:

$$Q_m(x_1, x_2, \ldots, x_{m-1}) := \frac{m-1}{2}\left(\sum_{j=1}^{m-1} x_j^2\right) - \sum_{1\leq j<k\leq m-1} x_j x_k.$$

We want to determine the values represented by this form when our inputs come from the Apéry set of a semigroup. A sufficient condition is that $\max_{j\neq k} \frac{x_j}{x_k} \leq 2$. Setting $m = 5$ gives the problem from the beginning of the abstract. Our main result implies the following.

Corollary 1. *For each $n \geq 6$ there exists a numerical semigroup S with multiplicity 5 and $w(S) + g(S) = n$.*
For each $n \geq 6$ there exists a 5-core partition λ of n which is not a t-core partition for any $t < 5$ with the additional property that for any $j \in H(\lambda)$ there exists a box in the first column of the Young diagram of λ with hook number equal to j.

We describe some of the ingredients of the proof below. In order to impose the condition that (x_1, x_2, x_3, x_4) satisfies $\max_{j\neq k} \frac{x_j}{x_k} \leq 2$ we note that this vector is in this cone if and only if the absolute value of the cosine of the angle defined by this vector and $(1, 1, 1, 1)$ is large. We treat this scaled inner product as a variable and find a polynomial in this variable which is negative outside of the cone and has positive average on the unit sphere. Therefore, if we show that this average of this polynomial taken over all vectors of norm N is positive, then there exists a vector of norm N in our cone.

We express this polynomial as a linear combination of Chebyshev polynomials of the second kind, which lead to closely related harmonic polynomials. These harmonic polynomials give weighted theta functions. In this setting, the theta function weighted by the harmonic polynomial of degree $2n$ is a modular form of level 5, weight $2 + 2n$, and character χ, the Legendre character mod 5.

One of these weighted theta functions is the Eisenstein series for our space and the other nonzero ones are cusp forms. We can determine the q-expansion of the Eisenstein series explicitly and note that the q^n coefficient is $\gg n^{1-\epsilon}$. We use a theorem of Deligne to bound the contribution from the cusp forms. Combining these bounds with the expansion described above shows that the average of this polynomial taken over vectors of norm N must be positive if N is squarefree and either N has 9 or more prime factors, or N has a prime factor at least 5471. We now have a large but finite list of squarefree N to consider.

We generate a list of candidates and for each one determine the relevant cusp form coefficients and the corresponding term of the overall q-expansion. This completes our analysis for squarefree N. It is not difficult to extend these ideas to general N. We show that the only N for which the average of the polynomial taken over vectors of norm N is negative are $\{1, 2, 3, 4, 6, 8, 10, 14\}$. We can give explicit representatives of vectors of norm N in our cone for some of these values, completing the proof of the main theorem.

References

[1] M. Bras-Amorós and A. de Mier, Representation of numerical semigroups by Dyck paths. Semigroup Forum 75 (2007), no. 3, 677-682.

[2] A. Granville and K. Ono. Defect zero p-blocks for finite simple groups. Trans. Amer. Math. Soc., 348 (1996), no. 1, 331-347.