Math 232a: Algebraic Number Theory

Fall 2017 Course Information and Syllabus
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Lectures: M,W,F 12:00 - 12:50 in Rowland Hall 340N.
Office Hours: W 11:00 - 12:00, RH 540c.
Also, please feel free to email me to set up an appointment.

Course Overview

Math 232a is the first quarter of a year-long introduction to algebraic number theory. One of the main goals of number theory is to understand solutions to Diophantine equations. For example:

- What are all the integer solutions to $x^2 - dy^2 = 1$?
- Which primes can be written as the sum of two squares?
- What are all the integer solutions to $y^2 = x^3 - 2$?

All of these questions can be phrased in terms of the arithmetic of number fields. A *number field* K is a finite extension of $\mathbb{Q}$. The field K contains its *ring of integers* $\mathcal{O}_K$, which plays a role analogous to $\mathbb{Z}$ inside of $\mathbb{Q}$. It is not generally true that $\mathcal{O}_K$ is a unique factorization domain or a principal ideal domain, but it does have a kind of unique factorization of ideals into products of prime ideals.

One of the main goals of this course will be to set up a vocabulary for talking about the arithmetic of number fields. We will study the *ideal class group*, a finite abelian group that measures the failure of unique factorization. We will also prove *Dirichlet's unit theorem*, which describes the structure of the group of units in algebraic number rings.

We will emphasize examples and explicit calculations throughout the course, focusing on *quadratic fields* and *cyclotomic fields*. This course will serve as a foundation for the next two quarters of the 232 sequence in which we will discuss some of the following topics: Local fields, class field theory, quadratic forms, zeta functions, and modular forms.

Major Topics

1. The ring of integers of a number field. Traces, norms, and discriminants.
2. Dedekind domains. Factorization of ideals into products of prime ideals.
3. The ideal class group and the geometry of numbers.
4. Dirichlet's unit theorem.
5. Factoring of prime ideals in extensions.
6. Examples: Quadratic Fields and Cyclotomic Fields.

Course Texts

1. *Number Fields* by Daniel Marcus.
   This book provides a concrete introduction to number fields with many excellent examples and exercises.

2. *Number Rings*: Course Notes by Peter Stevenhagen.
   Available online: `http://websites.math.leidenuniv.nl/algebra/ant.pdf`.
   These notes cover much of the same material as Marcus' book from a slightly more advanced perspective. These notes also include topics that are not covered in Marcus, for example localization and class groups of orders.

3. *Algebraic Number Theory*: Course notes by Matt Baker.
   Available online: `http://people.math.gatech.edu/~mbaker/pdf/ANTBook.pdf`.
   There notes cover similar material to the main sources given above. Lots of nice examples, applications, and exercises.

Prerequisites

The main prerequisite for this course is a good graduate-level course in algebra. We will need some concepts about rings, modules, Galois theory, and commutative algebra throughout the course. If you are nervous about your level of algebraic preparation, several of the texts I suggested have sections on algebraic background. I would highly recommend reading all three appendices from Marcus, and Chapter 1 of Milne's Algebraic Number Theory notes by the end of the first week of lecture.

Grading

- Weekly Homework: 70%
- In-Class Exam (Monday, November 6): 15%
- Final Problem Set (Due Friday, December 15th): 15%

Weekly homework assignments will be a very important part of the course. The best way to absorb the key concepts from this course will be to see lots of examples and do lots of problems. I encourage you to work together on these problem sets. You can use any outside resources (other textbooks, papers, the internet, etc.) but if you find a solution somewhere you must acknowledge it.

We will also have one in-class exam. I do not want it to be too difficult- the purpose of the exam is to check that you understand the main definitions and can work with basic examples.

Instead of a final exam we will have a final problem set. It will be like a regular problem set, but longer, and it should be done individually.