**Math 244: Discrete Mathematics**
**Final Lecture Summary, Fall 2014.**

# 1 Lecture 1

In Lecture 1 we discussed several motivating problems related to material that we will cover in the course. These included a problem about flipping coins, the problem about students randomly grabbing lunchboxes, the problem of the 100 prisoners and the 100 names, the birthday problem, the 'Three Houses, Three Resources' problem, Bridges of Königsberg, the Traveling Salesman Problem, Knight's Tours and Hamiltonian Cycles, Mastermind, and the MTV game show "Are You the One?"

# 2 Lecture 2

In this lecture we discussed the basics of counting in terms of words of length $k$ from an alphabet of $n$ letters. We talked about counting sequences (where the order matters) both with and without repetition, and counting sets (where the order doesn't matter) with and without repetition. We talked about the concept of a multiset, a set that is allowed to have repeated elements. We talked about how coin-flipping problems can be phrased in this setup of words made up of two letters- H and T. We also saw that lots of discrete probability problems (for example, the coin ones) are really problems about counting the sizes of two different finite sets. We defined the binomial coefficients and showed that the sum of $\binom{n}{k}$ over $k$ is $2^n$.

**Reading**: Chapter 1 of the LVP book is useful reading for these general counting problems. Section 3.2 and 3.4 also cover some of the counting we talked about.

# 3 Lecture 3

We talked about counting with repetition and explained the 'Stars and Bars' method of counting. We talked about the birthday problem. We also talked about compositions and weak-compositions and gave a preview of Pascal's Triangle.

**Reading**: Section 3.2 and 3.4 also cover this counting. Section 2.5 covers the Birthday Problem a little differently than we did.

# 4 Lecture 4

We started by talking about induction. I explained the main idea and then used $\sum_{k=1}^{n} k$ as an example. We started to discuss properties of binomial coefficients and some of the identities in Pascal's triangle. We gave some combinatorial proofs- for example that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. We also proved the Hockey-Stick identity.

**Reading**: Induction - Section 2.1 of LVP (also 1.6 of Matousek Nesetril). Binomial coefficients - Sections 3.5 and 3.6. At this point we've covered 3.1-3.3 of MN.

# 5 Lecture 5

We defined the Fibonacci numbers and showed how they appear as sums of shallow diagonals in Pascal's triangle. We also used induction to prove a formula for the Fibonacci numbers. We explained how the Fibonacci numbers count the number of ways of going up $n$ stairs either one or two at a time.

**Reading**: Chapter 4 of LVP covers Fibonacci numbers, but we will not cover the 'partial fractions' method of finding the correct formula.

# 6 Lecture 6

We gave several proofs of the identity involving the sum of the squares of the binomial coefficients, which is sometimes called Vandermonde's Identity. We discussed the Binomial Theorem and gave two proofs of it. We then gave a few applications, one of which involved complex numbers.

**Reading**: The Binomial Theorem is Section 3.1 (although there is no discussion of complex numbers).

# 7 Lecture 7

We used anagrams to introduce multinomial coefficients and then talked about the multinomial theorem. We then talked about some more counting problems focusing on questions about making two teams of $n$ out of $2n$ students.

**Reading**: Anagrams are Section 3.3 of LVP. The counting problems mostly reinforce material from Chapter 1.

## 8 Lecture 8

We focused on 'The Subtraction Principle' and discussed seating people around a table so that enemies can't sit next to each other and choosing 3 of 20 guests at a circular table no two of whom are neighbors. We saw that it is sometimes easier to count a set by counting the things not in the set. This is sometimes called the Subtraction Principle. We also discussed how to count lattice paths and how the Catalan numbers give a formula for those that do not cross the main diagonal line.

**Reading**: Unfortunately, Catalan numbers are not covered in the LVP book. There is a short discussion of them in the Matousek Nesetril book but it does not give the "Reflection Principle" proof that we saw. This is actually given on the Wikipedia entry for Catalan numbers, or in a number of other combinatorics textbooks/websites, including in the two links I sent out with an earlier homework.

## 9 Lecture 9

We talked about the Pigeonhole Principle and used the fact that there are two people in NYC with the same number of hairs on their heads as an example. We talked about some geometric applications of the pigeonhole idea and also gave the example involving 10-element subsets of $\{1, 2, \ldots, 100\}$.

We talked about inclusion/exclusion in terms of the three clubs within Math 244 and drew a Venn diagram showing the intersection between sets.

**Reading**: 2.4 of LVP for the Pigeonhole Principle, 2.3 of LVP for Inclusion-Exclusion (with the general statement and proof from MN 3.7).

## 10 Lecture 10

We stated the general version of the Inclusion/Exclusion Principle and gave a counting proof, showing that an element $x$ is counted exactly once by the formula no matter how many sets it is contained in. We talked about how to write permutations in one-line, two-line, and cycle notation. We solved the problem of $n$ students walking into a room with $n$ lunchbox and each grabbing a random lunch when they leave using Inclusion/Exclusion. This comes down to finding a formula for the number of permutations with $\{1, 2, \ldots, n\}$ with no fixed points, that is $x_1 x_2 \cdots x_n$ with no $i$ such that $x_i = i$. These are called derangements.

**Reading**: Section 3.8 of MN for counting derangements and Section 3.2 for cycle structure of permutations.

# 11 Lecture 11

We solved the problem of the 100 Prisoners and 100 Boxes by counting permutations that have no long cycles. At the end of lecture we talked about how the binomial coefficients grow. We noted that they grow as we move down Pascal's triangle and showed that they increase as we move across a given row until we reach the middle, and then decrease as we move back down. We showed that an integer $m > 1$ occurs in Pascal's triangle at most $2m$ times.

**Reading**: For the Prisoners problem, see these two references:
`https://math.dartmouth.edu/~pw/solutions.pdf`
`http://www.inference.phy.cam.ac.uk/itila/cycles.pdf`

# 12 Lecture 12

We started by discussing the size of the largest entry in the $2n$th row of Pascal's triangle, $\frac{2n}{n}$. We saw that we had upper and lower bounds that came from summing across the whole row,

$$\frac{2^{2n}}{2n+1} \leq \binom{2n}{n} \leq 2^{2n}.$$

We then considered several values of $\binom{2n}{n}/4^n$ and observed that it seems to go to zero.

We then talked about the size of sums like

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{k}.$$

We saw that when divided by the sum of the whole row, $2^n$, the limit seems to go to zero. We then stated the Law of Large Numbers, a more general quantitative version of this statement.

We introduced asymptotic notation and 'little oh' $f = o(g)$ notation. We gave some examples and basic facts. We stated that the sum of the first $n$ $m$th powers is asymptotic to $\frac{n^{m+1}}{m+1}$, which you will show on the homework.

**Reading**: This opens up the start of a new section of the course. Please read through Sections 3.7 and 3.8 of the Lovasz book. Chapter 5 gives the proof of the Law of Large Numbers. We will come back to this next week. These sections together cover the majority of the material for the next few lectures. Section 2.2 of Lovasz introduces asymptotic notation and mentions Stirling's Formula.

I think the Matousek textbook handles some of this material in a more extensive and rigorous way. See Section 3.4 for the material on asymptotic and little oh notation. There are lots of good examples in here. Note that I never introduced the 'big oh' notation.

## 13 Lecture 13

We proved the Arithmetic Mean - Geometric Mean inequality and used it to give estimates on the size of $n!$. We then used similar ideas to give upper and lower bounds for the size of $\binom{n}{k}$. We saw that these bounds did not really help us pin down the size of $\binom{2n}{n}$.

We then stated Stirling's Formula, an asymptotic formula for the size of $n!$. You do not need to know how to prove this, since it uses some advanced calculus ideas, but you should know the statement. We then used Stirling's formula to see that $\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$.

We then stated, but did not prove yet, two main propositions that go into the proof of the Law of Large Numbers. The first one gives good upper and lower bounds for the size of the ratio $\frac{\binom{2m}{m-t}}{\binom{2m}{m}}$. The next one gives a bound for the size of the first $m-t$ binomial coefficients in row $2m$ of Pascal's triangle. The bound is given in terms of this ratio described above.

**Reading**: See Section 3.5 of the Matousek book for estimates of the factorial function and section 3.6 for estimates on the sizes of binomial coefficients.

## 14 Lecture 14

We used the two propositions from the previous lecture to give a proof of the Law of Large Numbers. We first proved an intermediate result, giving an upper bound for the probability that out of $2m$ flips we get at most $m-t$ heads or at least $m+t$ heads. We then proved the two propositions from the previous lecture. One of these proofs involved a tricky idea- to estimate the size of a product of $t$ terms, we took the natural log of this product and then estimated the sizes of the natural log of each of the resulting $t$ terms in the sum.

We also talked about the idea of finding the 95% confidence interval for the number of heads in $2m$ coin flips. We know that a range of size $\epsilon \cdot (2m)$ is too big- as $m$ goes to infinity, the probability that the number of heads is off by at most $\epsilon m$ goes to zero for any fixed $\epsilon > 0$. We saw that with the bounds we have, a range of size approximately $4\sqrt{m}$ seemed to be of the right size.

**Reading**: This is the proof given in Chapter 5 of Lovasz. For more on estimating logarithms, see Section 2.5 of Lovasz.

## 15 Lecture 15

I introduced the idea of a one-dimensional random walk. That is, you line on the integer number line and start at 0. Every day you flip a coin. If it comes up Heads, you move one step to the right, and if it is Tails, you move one step left. We say that your average position after $2n$ days is 0 and then started to discuss how far away you expect to be after $2n$ days. We estimated this quantity by using the upper and lower bounds that went into the proof of the Law of Large Numbers.

I also introduced the Gambler's Ruin problem and we used similar ideas to say that after $n$ flips, no matter how much money you start with, as $n$ goes to infinity you will have 0 money with probability approaching 1.

I showed how the Catalan numbers appear in the one-dimensional random walk problem.

**Reading**: Section 12.6 of the MN book gives a very different view of random walks. This uses generating functions, which we will not discuss here. I sent out a reference that covers this random walk material, the Gambler's ruin, and much more.

## 16 Lecture 16

I defined a random variable for a finite space of possible outcomes. I then gave the definition of the expected value and variance of a random variable. I stated linear of expectation as a fact without proof. I also gave an example to show that variance does not have to be linear for random variables that are not independent, but stated that it is linear for independent random variables. We used all of this material to see why computing the expected square of the distance from the origin on a $2n$ step random walk is so much easier to compute than the expected absolute value of the distance.

**Reading**: Section 10.2 of the MN book has an introduction to finite probability spaces. Section 10.3 defines random variables and their expectation. These two sections cover much more material than I did in lecture. You do not need to be familiar with all of this for the exams, but I would definitely recommend that you read through these sections.

## 17 Lecture 17

I started lecture by stating a version of the Central Limit Theorem, one of the most important results in statistics. It basically says that the binomial distribution 'converges' to the normal distribution. That is, if you have a binomial random variable, you can compute the probability that it takes values in a certain range by computing an integral related to the normal distribution and expect the result to be very close as $n$ goes to infinity.

I introduced some of the basic notions of graph theory. I defined a graph, gave some basic examples, and defined some simple graph properties. We proved in two different ways that in any graph with an odd number of vertices, the number of vertices of even degree is odd.

**Reading for this week**: For graph theory we followed the LVP book pretty closely. The graph theory in the MN book is often written at a more abstract level, so it would be useful to look at it and get a second perspective on some of the material.

LVP section 7.1, 7.2, and 7.3. MN Sections 4.1, 4.2, 4.4.

## 18 Lecture 18

In this lecture we introduced treess and bipartite graphs. We proved a statement about what it means for a graph to be connected. We defined a walk on a graph, Eulerian walks and tours. We then stated the main theorem about Eulerian walks, and gave a major part of the proof, showing that if $v$ is a vertex of odd degree, then every Eulerian walk on $G$ either starts or ends at $v$.

## 19 Lecture 19

We began lecture by clearing up the definition of a path, a cycle, and a walk. We talked about what it means for a graph to 'contain a path'. We then proved the main theorem about Eulerian tours, that a connected graph contains one if and only if every vertex has even degree. We defined a Hamiltonian path and Hamiltonian cycle. We showed that if $G$ is a connected graph on $n > 2$ vertices and every vertex has degree at least $n/2$, then $G$ has a Hamiltonian cycle.

**Reading**: This proof is given in Section 7.3 of LVP, which also discusses Hamiltonian cycles (but does not give the proof we gave in lecture.) That result is called Dirac's Theorem, and this proof can be easily found on many websites.

## 20 Lecture 20

We defined the concept of isomorphism of graphs and discussed the question of how many graphs there are. We then showed that a connected graph is bipartite if and only if it contains no cycles of odd length.

**Reading**: The main result about bipartite graphs is the subject of Section 13.2 of LVP. This entire chapter is about graph coloring problems and might be interesting to look at (but will not be necessary for the final exam.) Isomorphisms of graphs are covered in detail in Section 4.1 of MN.

## 21 Lecture 21

We defined the chromatic number of a graph and gave some basic examples. We then started to discuss trees in more detail. We proved several properties equivalent to being a tree. We described the tree-growing procedure and showed that it produces all trees.

**Reading**: Trees are discussed in Section 8.1 of LVP and the tree-growing procedure is covered in section 8.2 The equivalent characterizations of trees are covered in detail in Section 5.1 of MN.

## 22 Lecture 22

We will show that every connected graph contains a spanning tree. We will then talk about graphs with edge costs and explain Kruskal's algorithm for finding the cheapest spanning tree. We will then talk about the Traveling Salesman Problem. **Reading**: Spanning trees and Kruskal's algorithm are covered in Sections 9.1 of LVP and 5.3 and 5.4 of MN.

## 23 Lecture 23

We discussed how to use the cheapest spanning tree to quickly find a tour that costs at most twice the price of the best possible tour.

We then proved Euler's formula for graphs. We showed that $K_5$ is not planar. I explained this in more detail in a follow-up email. Very similar ideas, also explained in that email, show that $K_{3,3}$ is not planar and that every planar graph has a vertex of degree at most 5.

**Reading**: The application to the Traveling Salesman is the subject of Section 9.2 of LVP. Euler's formula is Sections 12.1 and 12.2 of LVP. This material is covered in a very different way (the introduction to it at least) in Sections 6.1 - 6.3 in MN.