Math 206A: Algebra

Final Exam

Thursday, December 17, 2020.

- You have **2 hours** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (10 Points)	
2 (9 Points)	
3 (9 Points)	
4 (10 Points)	
5 (10 Points)	
6 (5 Points)	
Total	

Problems	
7 (12 Points)	
8 (6 Points)	
9 (10 Points)	
10 (12 Points)	
11 (5 Points)	
Total	

Problems

1. (a) Define a field.
   (b) Define an integral domain.
   (c) Prove that a finite integral domain is a field.

2. Decide which of the following are subrings of $\mathbb{Q}$. Give a brief justification for your answer.
   (a) The set of nonnegative rational numbers.
   (b) The set of all rational numbers with odd numerators (when written in lowest terms).
   (c) The set of all rational numbers with even numerators (when written in lowest terms).

   Note: For this question we are using Dummit and Foote's definition of a subring. That is, a subring does not necessarily have to contain an identity.

3. Decide which of the following are ideals of $\mathbb{Z}[x]$:
   (a) The set of all polynomials whose coefficient of x^2 is a multiple of 3.
   (b) The set of all polynomials whose constant term, coefficient of x, and coefficient of x^2 are zero.
   (c) The set of all polynomials whose coefficients sum to zero.

4. Find all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}/30\mathbb{Z}$. Explain how you know your list is complete.

   Note: For this question we are using Dummit and Foote's definition of a ring homomorphism. That is, a ring homomorphism $\varphi\colon R \to S$ between rings with identities does not necessarily have to take the identity of R to the identity of S.

5. (a) State the Orbit-Stabilizer Theorem.
   (b) Let G be a finite p-group acting on a finite set X. Prove that

$$|X| \equiv \#\{\text{Fixed points of this action}\} \pmod{p}.$$

6. Does there exist a group G where $G \times G$ contains an element of order 15, but G does not contain an element of order 15?
   Either give an example of such a G or prove that such an example does not exist.

7. Let $p < q$ be odd primes. Let G be a group of order $2pq$.
   (a) Prove that G is not simple.
   (b) Define what it means for a group to be solvable.
   (c) Prove that G is solvable.

8. (a) Describe the conjugacy classes of S_4.
   (b) How many elements are in each conjugacy class?

9. (a) Prove that a subgroup of a cyclic group is cyclic.
   (b) Is the automorphism group of a cyclic group necessarily cyclic? Explain your answer.

10. Let G be a group of order 42.
    (a) Prove that G has a subgroup of order 6.
    (b) Prove G has a subgroup of order 21.
    (c) Prove that G is isomorphic to a semidirect product of two nontrivial groups.

11. Either prove the following statement or give a counterexample.
    For any group G, the map $\varphi\colon G \to G$ defined by $\varphi(g) = g^2$ is a homomorphism.