Math 206A: Algebra
Midterm 2: Solutions
Monday, November 23, 2020.

Solutions

1. Suppose G is an abelian group, and H_1, H_2 are subgroups.
Either prove the following statement or find a counterexample.

$$\text{If } G/H_1 \cong G/H_2, \quad \text{then } H_1 \cong H_2.$$

Solution: This is false. Let $G \cong \mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let $H_1 \cong \mathbb{Z}/p^2 \times \{0\}$ and $H_2 \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Just by counting we see that $|G/H_1| = |G/H_2| = p$, so $G/H_1 \cong G/H_2$ are isomorphic even though $H_1 \not\cong H_2$.

2. State whether the following statement is true or false.
Give a 1-2 sentence explanation for your answer.

$$\text{Every finite subgroup of } \mathrm{GL}_n(\mathbb{Q}) \text{ is abelian.}$$

Solution: This is false. In lecture we have seen that identifying elements of S_n with permutation matrices gives a subgroup of $\mathrm{GL}_n(\mathbb{R})$ isomorphic to S_n. In fact, all of these matrices have entries equal to 1 or 0, so we see that $\mathrm{GL}_n(\mathbb{Q})$ has a subgroup isomorphic to S_n.

3. (a) Describe all abelian groups of order 64 up to isomorphism.
(That is, give a list of abelian groups of order 64 such that every abelian group of order 32 is isomorphic to one in your list and no two of these groups are isomorphic to each other.)
Solution: By the fundamental theorem for finite abelian groups, there is a bijection between isomorphism classes of groups of order $64 = 2^6$ and partitions λ of 6. We see that every group of order 64 is isomorphic to one in the following list:

$$\begin{array}{rl}
\mathbb{Z}/2^6\mathbb{Z}, & \mathbb{Z}/2^5\mathbb{Z} \times \mathbb{Z}/2^1\mathbb{Z},\ \mathbb{Z}/2^4\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z},\ \mathbb{Z}/2^4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/2^3\mathbb{Z}, & \mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ \mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z}, & \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \\
\mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}
\end{array}$$

(b) Is $\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/48\mathbb{Z}$ isomorphic to $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/60\mathbb{Z}$?
Prove your answer is correct.

Solution: These groups are not isomorphic. We write them in terms of their elementary divisor decompositions. We do this by looking at the largest powers of p dividing each of the factors and then rearranging:

$$\begin{aligned}
\mathbb{Z}/30\mathbb{Z} \times \mathbb{Z}/48\mathbb{Z} &\cong (\mathbb{Z}/2^4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}/5\mathbb{Z}, \\
\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/60\mathbb{Z} &\cong (\mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/2^2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}/5\mathbb{Z}.
\end{aligned}$$

Since the 2-parts of these groups are not isomorphic, the groups are not isomorphic.

4. Let H, K be two subgroups of a finite group G such that $H \cap K = \{1\}$ and $|H| \cdot |K| = |G|$. Does it follow that $G \cong H \times K$?
Either prove this statement or give a counterexample.

   Solution: It is not always true that $G \cong H \times K$. The Recognition Theorem for Direct Products also requires that both H and K are normal subgroups in G. For example, take $G = D_{2n},\ H = \langle r \rangle$ and $K = \langle s \rangle$. We know that

$$D_{2n} \ncong \langle r \rangle \times \langle s \rangle \cong \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

5. Let $\varphi\colon G \to H$ be a surjective homomorphism between finite groups.
Prove that the image of a Sylow p-subgroup in G is a Sylow p-subgroup in H.

   Solution: Let P be a Sylow p-subgroup of G. A big idea here is to show first that $\varphi(P) \leq H$ is a p-group. Here are two ideas for how to do this:

   (a) If $x \in G$, then $|\varphi(x)|$ divides $|x|$, so every element in $\varphi(P)$ has order equal to a power of p. Therefore $|\varphi(P)|$ is a power of p. (By Cauchy's Theorem, if $|\varphi(P)|$ was not a power of p it would have to have an element of order not equal to a power of p.)

   (b) Let $N = \ker(\varphi)$. We have

$$\varphi(P) = \{pN) \colon p \in P\} = PN/N.$$

   By the Second Isomorphism Theorem,

$$PN/N \cong P/(P \cap N),$$

   that is

$$\varphi(P) \cong P/\ker(\varphi_{|P}).$$

   (We can also see this by applying the 1st Isomorphism Theorem to the surjective homomorphism $\varphi_{|P}\colon P \to \varphi(P)$.)

   Now we do the rest in two different ways:

(a) Suppose $|G| = p^a \cdot m$ where $p \nmid m$. Since φ is surjective, the 1st Isomorphism Theorem says that $H \cong G/\ker(\varphi)$, so $|H|$ divides $|G|$. In particular, $|H| = p^b \cdot m'$ where $b \leq a$ and $p \mid m'$. This means that p^{a-b} is the highest power of p dividing $|\ker(\varphi)|$.

Suppose that $|\varphi(P)| = p^c < p^b$. Therefore,

$$p^c = |\varphi(P)| = |P|/|\ker(\varphi_{|_P})| = p^a/p^{a-c}.$$

We know that $\ker(\varphi_{|_P}) = P \cap \ker(\varphi) \leq \ker(\varphi)$. But we see that p^{a-c} must divide $\ker(\varphi_{|_P})$, while p^{a-b} is the largest power of p dividing $\ker(\varphi)$. This is a contradiction.

(b) Since P is a Sylow p-subgroup of G, $[G\colon P]$ is relatively prime to P. Therefore, $[G : PN] = [G : P]/[PN : P]$ is also relatively prime to p.

We see that

$$[H\colon \varphi(P)] = [G/N\colon PN/N] = [G : PN]$$

is relatively prime to P as well. Since $\varphi(P)$ is a p-group, we see that it is a Sylow p-subgroup of H.

6. A subgroup H of a group G is **characteristic** if for every $\sigma \in \mathrm{Aut}(G)$, $\sigma(H) = H$. Prove that every subgroup of a cyclic group is characteristic.

Solution: First suppose G has order n. A cyclic group of order n has a unique subgroup of order d for each d dividing n. If $\sigma \in \mathrm{Aut}(G)$ and $H \leq G$, then $\sigma(H) \cong H$. In particular, $|\sigma(H)| = |H|$. Since H is the only subgroup of order $|H$, we must have $\sigma(H) = H$.

Now suppose G is an infinite cyclic group. So $G \cong \mathbb{Z}$. If H is the trivial subgroup this is clear. Every other subgroup H of G has index n for some $n \geq 1$, and there is exactly one such subgroup for each n. If $\sigma \in \mathrm{Aut}(G)$ and $H \leq G$, then $\sigma(H) \cong H$. In particular, $[G\colon \sigma(H)] = [G\colon H]$. Since H is the only subgroup of G with index n, we must have $\sigma(H) = H$.

7. Let G be a group and let $[G, G]$ denote its commutator subgroup. Suppose that $H \leq G$ satisfies $[G, G] \leq H$. Prove that H is a normal subgroup of G.

Solution: We know that $xyx^{-1}y^{-1} \in H$ for all $x, y \in G$. Let $h \in H$ and $g \in G$. We need to show that $ghg^{-1} \in H$. Suppose $ghg^{-1} = x$. We know that $ghg^{-1}h^{-1} = xh^{-1} \in H$. Therefore, $xh^{-1} \cdot h = x \in H$ also.

Solution 2: G' is a normal subgroup of G and $G/[G, G]$ is abelian. Every subgroup of an abelian group is normal. $H/[G, G]$ is a subgroup of $G/[G, G]$, so $H/[G, G]$ is normal in $G/[G, G]$. By the Lattice Isomorphism Theorem (part (5) on page 99), H is normal in G.

8. Suppose that G is a group of order $351 = 3^3 \cdot 13$. Prove that G is not simple.

Solution: By Sylow III,

$$n_3 \equiv 1 \pmod 3, \quad \text{and} \quad n_3 \mid 13,$$
$$n_{13} \equiv 1 \pmod{13}, \quad \text{and} \quad n_{13} \mid 27.$$

So $n_3 \in \{1, 13\}$ and $n_{13} \in \{1, 27\}$.

So if G is simple, then $n_3 = 13$ and $n_{13} = 27$. But if $n_{13} = 27$ then G contains $27 \cdot (13-1) = 324$ element of order 13. There are only 27 elements of order not equal to 13. Since a Sylow 3-subgroup does not contain any elements of order 13, any Sylow 3-subgroup must be a subset of these 27 elements. Since a Sylow 3-subgroup of G has order 27, these 27 elements must be a Sylow 3-subgroup, and this Sylow 3-subgroup is unique.

9. Give an example of an infinite group G in which every element of G has finite order.
No explanation is needed– you just need to give the example.

Solution: Consider $\prod \mathbb{Z}/2\mathbb{Z}$, the direct product of copies of $\mathbb{Z}/2\mathbb{Z}$, one for each positive integer i. This is a group. Every non-identity element of this group has order 2 since

$$(a_1, a_2, \ldots) + (a_1, a_2, \ldots) = (a_1 + a_1, a_2 + a_2, \ldots) = (0, 0, \ldots).$$

10. (a) Prove that Q_8 is not isomorphic to a semidirect product of two groups of order smaller than 8.

Solution: If $Q_8 \cong H \rtimes_\varphi K$ for some groups H, K of order smaller than 8, then Q_8 contains a normal subgroup $\widetilde{H}$ isomorphic to H and a subgroup $\widetilde{K}$ isomorphic to K such that $\widetilde{H}\widetilde{K} = Q_8$ and $\widetilde{H} \cap \widetilde{K} = \{1\}$. If $\widetilde{H}\widetilde{K} = Q_8$ and $\widetilde{H} \cap \widetilde{K} = \{1\}$, then

$$|\widetilde{H}| \cdot |\widetilde{K}| = |H| \cdot |K| = 8.$$

So if both $|H|, |K| < 8$, then also $|H|, |K| > 1$.
Every non-identity subgroup of Q_8 contains the subgroup $\langle -1 \rangle$. Therefore, $\langle -1 \rangle \in \widetilde{H} \cap \widetilde{K}$.
So $Q_8 \ncong H \rtimes_\varphi K$.

(b) Is D_8 isomorphic to a semidirect product of two groups of order smaller than 8?
If so, then give groups H, K and a homomorphism φ such that $D_8 \cong H \rtimes_\varphi K$.
If not, prove the D_8 is not isomorphic to a semidirect product of two groups of order smaller than 8.

Solution: We have seen that $D_{2n} \cong \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$, where $\varphi \colon K \to \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ is defined by setting $\varphi_x(h) = h^{-1}$ where x is the non-identity element of $\mathbb{Z}/2\mathbb{Z}$.