Math 206A: Algebra
Final Exam Practice Problems: Rings

The goal of this document is to provide you with some practice problems about rings that will help you get comfortable with the material that you might see on the Final Exam. We first give some exercise from Dummit and Foote and then give some problems from previous Algebra Comprehensive and Qualifying Exams. At the end, we include some additional problems that are good to think about, but I would not prioritize solving them unless you have extra time.

In these problems you should use the definitions for rings given in Dummit and Foote. For example, a ring R does not necessarily have an identity, and a ring homomorphism between two rings with identity does not necessarily take the identity of the first ring to the identity of the second ring.

Exercises from Dummit and Foote

1. Exercise 1 Section 7.3: Prove that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.
2. Exercise 2 Section 7.3: Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.
3. Exercise 5 Section 7.3: Describe all ring homomorphisms from the ring $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. In each case describe the kernel and the image.
4. Exercise 10 Section 7.3 (only some parts): Decide which of the following are ideals of $\mathbb{Z}[x]$. Give a brief justification for your answer.
   (a) The set of all polynomials whose constant term is a multiple of 3.
   (b) $\mathbb{Z}[x^2]$ (i.e., the polynomials in which only even powers of x appear).
   (c) The set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.
5. Exercise 20 Section 7.3: Let I be an ideal of R and S be a subring of R. Prove that $I \cap S$ is an ideal of S. Show by example that not every ideal of a subring S of a ring R need be of the form $I \cap S$ for some ideal of R.

1. If R is a commutative ring with a unit element and I is an ideal of R, when is R/I a field? Fully justify your answer.

2. Prove that a ring R is an integral domain if and only if the ideal $\langle 0 \rangle$ is prime.

3. Prove that a commutative ring R with identity is a field if and only if R has no nonzero proper ideals.

4. Let A and B be ideals of a commutative ring R.

   (a) Show that $A \cap B$ and $A + B$ are ideals of R.

   (b) If $R = \mathbb{Z}$, $A = (m)$, and $B = (n)$, describe the ideals $A \cap B$ and $A + B$.

5. In the ring $\mathbb{Z}$ find a single generator for each of the following ideals:

   (a) $\langle a \rangle \cap \langle b \rangle$,

   (b) $\langle a \rangle \cdot \langle b \rangle$,

   (c) $\langle a \rangle + \langle b \rangle$.

6. Suppose R is a commutative ring with identity $1_R \neq 0_R$.

   (a) What does it mean to say that R is an integral domain?

   (b) Prove that if R is a finite integral domain, then R is a field.

7. (a) Let R be an integral domain. Prove that $R[x]^\times = R^\times$, that is, the units in $R[x]$ are the constant polynomials whose constant term is a unit in R.

   (b) Find an example of a ring R and nonconstant polynomials $f(x), g(x) \in R[x]$ such that $f(x)g(x) = 1$.

8. Give an example of a ring R and

   (a) an ideal of R that is not principal;

   (b) an ideal of R that is prime but not maximal.

   Note: You should use the same ring R for each of the two examples above.

9. For each of the following rings find all of the maximal ideals (and fully justify your answer):

   (a) $\mathbb{Z}$,

   (b) $\mathbb{Z}/n\mathbb{Z}$.

10. Show that every maximal ideal in a commutative ring with a unit element is prime.

Additional Problems

The problems below are good ones to think about, but I think it makes sense to first prioritize really understanding the ones above if you don't have time to work through them all.

1. Exercise 24 Section 7.3: Let R be a ring with identity $1 \neq 0$ and let $\varphi\colon R \to S$ be a ring homomorphism.
   (a) Prove that if J is an ideal of S then $\varphi^{-1}(J)$ is an ideal of R. Apply this to the special case where R is a subring of S and φ is the inclusion homomorphism to deduce that if J is an ideal of S then $J \cap R$ is an ideal of R.
   (b) Prove that if φ is surjective and I is an ideal of R then $\varphi(I)$ is an ideal of S. Give an example where this fails if φ is not surjective.

2. Exercise 13 Section 7.4: (This exercise is similar to the previous one except that it focuses on prime and maximal ideals.) Let R be a ring with identity $1 \neq 0$ and let $\varphi\colon R \to S$ be a homomorphism of commutative rings.
   (a) Prove that if P is a prime ideal of S then either $\varphi^{-1}(P) = R$ or $\varphi^{-1}(P)$ is a prime ideal of R. Apply this to the special case where R is a subring of S and φ is the inclusion homomorphism to deduce that if P is an ideal of S then either $P \cap R = R$ or $P \cap R$ is a prime ideal of R.
   (b) Prove that if M is a maximal ideal of S and φ is surjective then $\varphi^{-1}(M)$ is a maximal ideal of R. Give an example where this fails if φ is not surjective.

3. Exercise 14 of Section 7.4 (Parts abcd): This exercise is a little long so I won't write it all out. The idea is to write down representatives for the elements of the quotient of a polynomial ring by an ideal generated by a monic polynomial. We then consider some nilpotent elements in polynomial rings.

Comprehensive and Qualifying Exam Problems

1. Let p be a prime and let R be the ring of all 2×2 matrices of the form
$$\begin{pmatrix} a & b \\ pb & a \end{pmatrix},$$
   where a, b are integers. Prove that R is isomorphic to the ring $\mathbb{Z}[\sqrt{p}]$.

   Note: This problem is very close to part of Exercise 12 of Section 7.3 of Dummit and Foote. If you have additional time, try the rest of the exercise as well.

2. Let R be a commutative ring with identity $1_R \neq 0_R$. Show that R contains a minimal prime ideal P, that is, a prime ideal which contains no smaller prime ideal.

   Note: This is very close to Exercise 36 in Section 7.4, so if you have additional time, try that one too.

3. Let R be a commutative ring with identity and with a unique maximal ideal M. Show that every nonunit $x \in R$ is in M.

   Note: This is part of Exercise 37 in Section 7.4. If you have additional time, try to solve that one too.

4. Suppose that R is a commutative ring with 1 and $x \in R$ lies in every maximal ideal of R. Prove that $1 + x$ is a unit in R.