Math 206A: Algebra Qualifying Exam Problems: Groups

In this first section we give several problems that are in the style of 'show there is no simple group of order n'. I recommend trying a few of these out to understand how these arguments typically work. (I don't think you need to solve all of them one right after the other!)

1. Fall 2017 #2: Prove that no group of order 150 is simple.
2. Spring 2017 #1: Let G be a group of order 80. Prove that G is not simple.
3. Spring 2016 #3: Prove that there is no simple group of order 520.
4. Spring 2015 #6: Suppose that $p < q$ are prime numbers.
   Prove that no group of order p^2q is simple.
5. Fall 2014 #4: Prove that no group of order 132 is simple.
6. Fall 2008 #3: Prove that there are no simple groups of order 30.
7. Spring 2007 #3: Prove that there are no simple groups of order 105.
8. Fall 2004 #3: Let G be a finite group of order $n > 2$. Let H be a subgroup of G such that $r = [G\colon H] > 1$. Assume that $r! < 2n$. Prove that G is no a simple group.
   Hint: Construct a map from G into S_r.

In this second section we give a whole bunch of problems that involve proving some classification result about groups of order n. As in the previous set of problem, I think you should solve some of these to get used to the kinds of arguments that come up, but I do not recommend solving them all in detail (unless you have lots of extra time).

1. Fall 2019 #2: Let G be a finite group of order p^2q where $p < q$ are primes.
   Prove that either G has a normal Sylow q-subgroup of G is isomorphic to A_4.
2. Spring 2018 #1: Classify all groups of order $2018 = 2 \cdot 1009$ up to isomorphism. Justify your answers. (You can assume that 1009 is a prime number.)
3. Fall 2015 #4: Let G be a group of order 70.
   Prove that G has a normal subgroup of order 35.
4. Spring 2011 #5: Prove that if G is a group of order $5 \cdot 7 \cdot 11$, then the center of G has order divisible by 7.
5. Spring 2010 #1: Classify all groups of order 44 up to isomorphism. Make clear which of them are abelian.
6. Fall 2009 #1: Prove that there are precisely four groups of order 28 up to isomorphism. How many of them are non-abelian?

7. Fall 2008 #9: Suppose p is an odd prime. Show that there are exactly 5 groups of order $2p^2$ up to isomorphism.

8. Fall 2007 #10: Classify all groups of order 6 up to isomorphism.

9. Spring 2006 #4: Prove that every group of order 185 is abelian. How many groups of order 185 are there, up to isomorphism?

10. Spring 2005 #5: Classify the groups of order 12, up to isomorphism.

The following problems ask you to prove something about a particular group.

1. Fall 2018 #10A: Describe the conjugacy classes of Q_8.

2. Spring 2018 #10A: Describe the conjugacy classes of A_4.

3. Spring 2017 #5: Let D_8 be the dihedral group of order 8.

   (a) Compute $Z(D_8)$.

   (b) Compute the commutator subgroup $[D_8, D_8]$.

   (c) Compute the conjugacy classes of D_8.

4. Spring 2017 #2: Prove that the additive group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the multiplicative group $\{z\colon \mathbb{C}\colon |z| = 1\}$.

5. Spring 2015 #5: Let $D_{2n} = \langle r, s\colon r^n = s^2 = 1,\ rs = sr^{-1}\rangle$.

   (a) Prove that every subgroup of $\langle r\rangle$ is normal in G.

   (b) If $n = 2m$ with m odd, prove that $D_{2n} = D_{4m} \cong \mathbb{Z}/2\mathbb{Z} \times D_{2m}$.

   (c) Is $D_{24} \cong \mathbb{Z}/3\mathbb{Z} \times D_{12}$?

6. Fall 2013 #2:

   (a) Describe all automorphisms of the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. How many are there?

   (b) Describe all automorphisms of the ring $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. How many are there?

7. Fall 2012 #9D: (Short Answer) What is the largest order of an element in D_{64}?

8. Fall 2012 #10 (parts B,D,E): For each of the following, either give an example or explain **briefly** why no such example exists:

   (a) A nonabelian group in which all the proper subgroups are cyclic.

   (b) A nonabelian group with trivial automorphism group.

   (c) An element of order 4 in $\mathbb{R}/\mathbb{Z}$.

9. Spring 2012 #1B: Recall that the exponent of a group G is the smallest positive integer n such that $g^n = 1$ for all $g \in G$. Compute the exponent of S_5.

10. Spring 2012 #9B: True/False: The group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ has exactly 2 subgroups of index 2.

11. Spring 2012 #9C: True/False (give a brief explanation): The group S_4 is solvable.

12. Spring 2009 #1:

    (a) Suppose p is an odd prime dividing n. Prove that a Sylow p-subgroup of D_{2n} is normal and cyclic.

    (b) Prove that if $2n = 2^\alpha \cdot k$ where k is odd, then the number of Sylow 2-subgroups of D_{2n} is k. Describe all these subgroups.

13. Fall 2008 #2 (parts A,C): For each of the following groups G, compute the number of subgroups of G (including the trivial subgroup).

    (a) G is a cyclic group of order 63.

    (b) $G = D_8$.

14. Spring 2007 #2 (First part): Show that the order of $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$ is

$$p^{n(n-1)/2} \prod_{i=2}^{n} (p^i - 1).$$

15. Spring 2008 #1A: Suppose G is a cyclic group of order 20. How many automorphisms does G have?

16. Fall 2006 #5: Recall that $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})/Z(\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}))$.

    (a) Prove that $Z(\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})) = \{\pm I_3\}$ where I_2 is the 2×2 identity matrix.

    (b) Prove that $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z}) \cong S_4$.

17. Spring 2005 #1: Let $\mathbb{C}^*$ be the group of nonzero complex numbers under multiplication. Let H_n be the subgroup of n^{th} roots of unity. Show that the quotient $\mathbb{C}^*/H_n$ is isomorphic to $\mathbb{C}^*$ by giving an explicit isomorphism.

Here are some additional problems that do not fit nicely into one of the categories above.

1. Spring 2019 #2: Suppose A is a finitely generated abelian group, B is a subgroup of A and $C = A/B$. Prove that if C is torsion free then the isomorphism classes of B and C determine the isomorphism class of A uniquely. Give a counterexample that shows that the isomorphism class of A may not be uniquely determined if C has non-trivial torsion.

2. Spring 2018 #2: Let P be a group of order $|P| = p^r$ for some prime p.

    (a) Prove that $Z(P) \neq 1$.

    (b) Prove that P is solvable.

3. Fall 2017 #9C: Indicate whether the following statement is true or false and give a brief justification: The center of a non-abelian group G is always properly contained in some abelian subgroup.

4. Fall 2015 #8: Suppose that H is a normal subgroup of a finite group G.

   (a) Prove or disprove: If H has order 2, then H is a subgroup of the center of G.

   (b) Prove or disprove: If H has order 3, then H is a subgroup of the center of G.

5. Fall 2012 #9 (parts A, C): State whether each statement is true or false and give a brief explanation.

   (a) If a group has an element of order m and an element of order n, then it has an element of order $\text{lcm}(m, n)$.

   (b) There are at most $(n!)^n$ groups of order n up to isomorphism.

6. Spring 2012 #3: Show that a group with exactly 3 elements of order 2 is not simple.

7. Spring 2009 #2: Let G be a group such that $\text{Aut}(G)$ is cyclic. Prove that G is abelian.

8. Fall 2008 #5: Suppose that G_1 and G_2 are finite groups, and $\gcd(|G_1|, |G_2|) = 1$.

   (a) Prove that if H is a subgroup of $G_1 \times G_2$ then there are subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$ such that $H = H_1 \times H_2$.

   (b) Give an example to show that the conclusion in the previous part is false if we do not require $\gcd(|G_1|, |G_2|) = 1$.

9. Fall 2008 #6: Suppose that G is a finite group and suppose that H is a nontrivial subgroup contained in every nontrivial subgroup of G.

   (a) Prove that the order of G is a power of some prime p and G has exactly $p - 1$ elements of order p.

   (b) Give an example of such a G and H where G is nonabelian of order 8.

10. Spring 2008 #1B: How many homomorphisms are there from $\mathbb{Z}$ to S_n? Explain your answer.

11. Spring 2008 #1C: If G is a group and $g \in G$ is an element of order 25, what is the order of g^{10}?

12. Fall 2006 #4: Suppose G is a group and H is a finite normal subgroup of G. If G/H has an element of order n, prove that G has an element of order n.

13. Fall 2006 #8: Suppose that H and K are subgroups of a group G and suppose that H and K have finite index in G. Show that $H \cap K$ also has finite index in G.