Math 206A: Algebra Summary: Groups

In this document, we'll list of some of the highlights that we covered in the first part of the course. This is meant to be a guide to help you study for the Final Exam. We will include a separate second document focusing on rings.

1 Group Theory Basics: Subgroups and Quotient Groups

1.1 Definitions

1. Group;
2. Subgroup;
3. Abelian Group;
4. Cyclic Group;
5. Subgroup Generated by a Subset;
6. Group Homomorphism;
7. Group Isomorphism;
8. Direct Product of Groups;
9. Quotient Group;
10. Normal Subgroup;
11. Simple Group;
12. Solvable Group;
13. Cycle Decomposition of a Permutation;
14. Sign of a Permutation;
15. Lattice of Subgroups of a Group;
16. Center of a Group;
17. Normalizer and Centralizer of a Subset of a Group;
18. The Conjugate of a set H by an element g;
19. Fibers of a Homomorphism;
20. Kernel of a Homomorphism;

21. Natural Projection Homomorphism;
22. Order of an Element;
23. Exponent of a Group;
24. Index of a Subgroup;
25. Product Set HK;
26. Conjugacy Class of an Element.

1.2 Examples of Groups

1. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$;
2. $(\mathbb{Q}^*, \cdot), (\mathbb{R}^*, \cdot), (\mathbb{C}^*, \cdot)$;
3. $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}^n$;
4. $\mathbb{Z}/n\mathbb{Z}$ (cyclic groups in general);
5. Dihedral Groups D_{2n};
6. Symmetric Groups S_n;
7. Alternating Groups A_n;
8. Matrix Groups $M_n(R), \mathrm{GL}_n(R), \mathrm{SL}_n(R)$, Affine Group, Heisenberg Group;
9. Quaternion Group Q_8.

1.3 Theorems

1. Subgroups and quotients of cyclic groups are cyclic.
2. $Z(G), C_G(A), N_G(A) \leq G$.
3. $N \trianglelefteq G$ if and only if $N = \ker \varphi$ for some group homomorphisms φ.
4. Lagrange's Theorem;
5. Cauchy's Theorem;
6. Isomorphism Theorems (1st, 2nd [Diamond], 3rd [Cancel like fractions one], 4th [Lattice Isomorphism Theorem]);
7. The index of subgroups is multiplicative.
8. The sign of a permutation is well-defined.
9. Subgroups and quotients of solvable groups are solvable.
10. S_n is not solvable for any $n \geq 5$.

2 Group Actions

2.1 Definitions

1. Left/Right action of a group G on a set X;
2. Kernel of a group action;
3. Faithful Group Action;
4. Free Group Action;
5. Orb_x for $x \in X$;
6. Stab_x for $x \in X$;
7. Fixed point of an action;
8. $\mathrm{Fix}_g(X)$ for $g \in G$.

2.2 Examples of Group Actions

1. G acting on itself by left multiplication;
2. G acting on itself by conjugation;
3. $H \leq G,\ G$ acting on G/H by left multiplication;
4. G acts on its subgroups by conjugation.

2.3 Theorems

1. Cayley's theorem.
2. There is a bijection between actions of a group G on a set X and group homomorphisms $G \to \mathrm{Sym}(X)$.
3. Let G act on X.
   (a) Different orbits are disjoint.
   (b) For each $x \in X,\ \mathrm{Stab}_x \leq G$ and $\mathrm{Stab}_{g \cdot x} = g\,\mathrm{Stab}_x\,g^{-1}$.
   (c) $g \cdot x = g' \cdot x$ if and only if g and g' lie in the same left coset of Stab_x. In particular, $|\,\mathrm{Orb}_x\,| = [G : \mathrm{Stab}_x]$.

4. Let G be a finite group acting on a finite set X and $x_1, \dots, x_r$ be representatives of the distinct orbits of this action.
$$|X| = \sum_i |\operatorname{Orb}_{x_i}| = \sum_i [G : \operatorname{Stab}_{x_i}].$$
Choose the particular action of conjugation and get Class Equation.

5. Burnside's Lemma.

6. Fixed Point Congruence.

7. Any nonabelian group of order 6 is isomorphic to S_3.

8. A finitely generated group G has only finitely many subgroups of index n for each $n \geq 1$.

9. Let G be finite and p be the smallest prime dividing $|G|$. Any subgroup of G of index p is normal.

10. Conjugacy classes in S_n are determined by cycle type.

11. A_5 is simple. A_n is simple for $n \geq 5$.

3 Automorphisms, Sylow's Theorem, Direct Products, and Semidirect Products

3.1 Definitions

1. Automorphism of a group;

2. Inner automorphism;

3. Characteristic subgroup;

4. Sylow p-subgroup;

5. Invariant factors and elementary divisors of a finite abelian group;

6. Semidirect product of H, K with respect to $\varphi \colon K \to \operatorname{Aut}(H)$.

3.2 Theorems

1. Sylow's Theorem (Parts I,II,III, and III*).

2. (a) Characteristic subgroups are normal.
   (b) If H is the unique subgroup of G of given order then H is characteristic.
   (c) Characteristic subgroups of normal subgroups are normal.

3. Let $H \trianglelefteq G$. Then G acts by conjugation on H as automorphisms of H. In particular, $G/C_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

4. A group of order 30 has a normal Sylow 3-subgroup and a normal Sylow 5-subgroup.

5. A group of order 12 has either a normal Sylow 3-subgroup or is isomorphic to A_4.

6. A group of order p^2q where p, q are distinct primes has either a normal Sylow p-subgroup of a normal Sylow q-subgroup.

7. Let p be a prime. An element of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ with order p is conjugate to a matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$. In this group $n_p = p + 1$.

8. Fundamental theorem of finitely generated abelian groups.

9. Primary decomposition theorem.

10. Recognition theorem for direct products.

11. Recognition theorem for semidirect products.

12. Classification of groups of order pq, 12 and 30.

13. Techniques to show that there are no simple groups of order n: Counting elements, ‘No subgroup of small index’.