Math 206B: Algebra
Homework 3

Due Friday, January 29th at 12:00PM.
Please email `nckaplan@math.uci.edu` with questions.

All exercises are from Dummit and Foote.

1. Exercise 1 of Section 8.3
   The third part of this exercise is a standard Algebra Qualifying Exam problem. Often it is asked as, "Determine all ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$".

2. Exercise 2 of Section 8.3
   On HW2 you solved two exercises about least common multiples. In this exercise, you see that in a UFD if you are given two elements that are factored into a product of irreducible elements it is straightforward to compute a least common multiple for them.

3. Exercise 4 of Section 8.3
   Sometimes it is easier to show that an equation has a rational solution than it is to show that it has an integer solution. Similarly, sometimes it is easier to show that an equation does not have an integer solution than it is to show that it does not have a rational solution. In this exercise, you see that for sums of two squares there is not a big different between finding solutions in $\mathbb{Q}$ and finding solutions in $\mathbb{Z}$.

4. Exercise 5 of Section 8.3
   For which positive squarefree integers D is the ring $\mathbb{Z}[\sqrt{-D}]$ a UFD? This is closely related to asking which quadratic integer rings with $D < 0$ are UFDs. In fact, when $-D \equiv 2, 3 \pmod 4$, these two questions are the same. There is a discussion of these questions above Exercise 8 of Section 8.1. (It turns out that a quadratic integer ring is a UFD iff it is a PID.) The result referred to there is known as the *Stark-Heegner theorem*. It is beyond the scope of this course, and is also more advanced than anything you would see in the graduate Number Theory course Math 232. This exercise solves part of this problem.

5. Exercise 6 of Section 8.3
   In this exercise you use what we know about prime/irreducible elements in $\mathbb{Z}[i]$ to understand the quotients $\mathbb{Z}[i]/I$ for certain ideals I.

6. Exercise 7 of Section 8.3
   This exercise builds on the previous one. In Exercise 10 of Section 8.1 you are asked to use the fact that $\mathbb{Z}[i]$ is a Euclidean domain to prove that $\mathbb{Z}[i]/I$ is finite for any nontrivial ideal I. Part (c) of this exercise gives another way to prove this finiteness.

7. **Suggested Extra Problem– you don't have to hand this is**: Exercise 8 of Section 8.3
   On HW2, you showed that several ideals in $\mathbb{Z}[\sqrt{-5}]$ are not principal and then showed that products of these ideals are principal. In particular, you showed that a certain product of these ideals gave the principal ideal generated by 6. In this exercise, you show that each of the ideals from this earlier exercise is prime. Therefore, you are seeing a factorization of the ideal (6) into a product of prime ideals. This is one instance of a more general phenomenon in quadratic integer rings that I talked about a little at the end of Lecture 7 Video 4.