Math 206B: Algebra
Midterm 2 Practice Problems

The goal of this document is to provide you with some practice problems for Midterm 2 from past Algebra Comprehensive and Qualifying Exams. I have made an attempt to divide up the problems by topic and also to indicate which ones we have already proven in lecture.

In these problems you should use the definitions for rings given in Dummit and Foote. For example, a ring R does not necessarily have an identity, and a ring homomorphism between two rings with identity does not necessarily take the identity of the first ring to the identity of the second ring.

Polynomial Rings

1. Algebra Comprehensive Exam Fall 2009: #2
   Let $\mathbb{Z}$ denote the ring of integers and consider the three commutative rings
   $$\begin{aligned} R_1 &= \mathbb{Z} \times \mathbb{Z} \\ R_2 &= \mathbb{Z}[\sqrt{5}] \\ R_3 &= \mathbb{Z}[x]/x^2\mathbb{Z}[x]. \end{aligned}$$
   For $1 \leq i < j \leq 3$ either prove that R_i and R_j are isomorphic rings, or prove that they are not.

2. Algebra Qualifying Exam Fall 2004: #6
   (a) Prove that the three additive groups $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Z}[x]/(x^2)$ are all isomorphic to each other.
   (b) Prove that no two of the rings $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}[i]$, and $\mathbb{Z}[x]/(x^2)$ are isomorphic to each other.

3. Algebra Qualifying Exam Fall 2017: #7
   Which of the following ideals of $\mathbb{Z}[x, y]$ are prime? Which are maximal? Justify your answer.
   $$(x, y),\ (x, 3y),\ (x^2 + 1, y),\ (x^2 + 1, 3, y),\ (x^2 + 1, 5, y).$$
   Note: Algebra Qualifying Exam Spring 2008 #4 and Algebra Qualifying Exam Spring 1997 #4 are both very similar to this.

4. Algebra Qualifying Exam Winter 2021: #3
   Let R be a PID. Suppose that $f, g, h \in R$ are such that $f = gh$ where g, h are relatively prime. Prove that
   $$R/(f) \cong R/(g) \times R/(h).$$
   Note: This is pretty similar to Proposition 16 in Section 9.5.

5. Algebra Qualifying Exam Winter 2021: #4
   Let R be an integral domain with field of fractions F. Assume that $p(x) \in R[x]$ is a monic polynomial and that it is possible to write $p(x)$ as a product
   $$p(x) = q(x)r(x)$$
   where $q(x), r(x) \in F[x]$ are monic polynomials of degree smaller than $\deg(p(x))$ and at least one of $q(x), r(x)$ is not in $R[x]$. Prove that R is not a UFD.

6. Algebra Qualifying Exam Fall 2020: #4
   Consider the ideal I of the polynomial ring $\mathbb{Z}[x]$ which is generated by a prime number p and a non-constant polynomial $f(x) \in \mathbb{Z}[x]$. Prove that I is maximal if and only if $f(x)$ is irreducible modulo p.

7. Algebra Qualifying Exam Fall 2013: #5
   Prove that for any field F and any positive integer n, the group $\mu_n(F) = \{x \in F : x^n = 1\}$ is a cyclic group.

8. Algebra Qualifying Exam Spring 2006: #6
   Determine the structure (as a direct product of cyclic groups) of the group of units of the ring $(\mathbb{Z}/5\mathbb{Z})[x]/(x^3 + 1)(\mathbb{Z}/5\mathbb{Z})[x]$.

9. Algebra Qualifying Exam Spring 2005 #10b
   True/False: The group of units in $\mathbb{Z}/12\mathbb{Z}$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Explain your answer.

10. Algebra Qualifying Exam Spring 2015: #7
    Determine the maximal ideals of the following rings. (Fully justify your answer.)

    (a) $\mathbb{Q}[x]/(x^2 - 5x + 6)$,
    (b) $\mathbb{Q}[x]/(x^2 + 4x + 6)$.

11. Algebra Qualifying Exam Spring 2009: #3

    (a) Determine whether the rings $(\mathbb{Z}/5\mathbb{Z})[x]/(x^2+1)$ and $(\mathbb{Z}/5\mathbb{Z})[x]/(x^2+2)$ are isomorphic.
    (b) List all ideals in the ring $\mathbb{Z}[x]/(2, x^3 + 1)$.

12. Algebra Qualifying Exam Fall 2012: #4
    List all the ideals in the ring $\mathbb{Q}[x]/(x^3 - x^2 - x + 1)$.

13. Algebra Qualifying Exam Fall 2001: #9

    (a) Find all maximal ideals of the ring $\mathbb{Q}[x]/(x^2 - 4)$.
    (b) Find all maximal ideals of the ring $\mathbb{Q}[x]/(x^2 - 1)$.
    (c) Express the ring $\mathbb{Q}[x]/(x^2 - 4)$ as a direct sum of fields.

Polynomial Rings: Irreducibility

1. Algebra Qualifying Exam Spring 2005: #9
   Suppose $f(x) \in \mathbb{Z}[x]$ is a polynomial of degree 5. Consider the following statements.

   (i) f has no roots in $\mathbb{Q}$;

   (ii) $f \equiv g_2 g_3 \pmod{11}$ where $g_2, g_3 \in (\mathbb{Z}/11\mathbb{Z})[x]$ are irreducible polynomials of degree 2 and 3, respectively.

   (iii) $f \equiv h_1 h_4 \pmod{17}$ where $h_1, h_4 \in (\mathbb{Z}/17\mathbb{Z})[x]$ are irreducible polynomials of degree 1 and 4, respectively.

   For each of the following assertions, either prove it is true or give a counterexample to show that it is false.

   (a) If (i) holds then f is irreducible in $\mathbb{Q}[x]$.

   (b) If (ii) holds then f is irreducible in $\mathbb{Q}[x]$.

   (c) If (iii) holds then f is irreducible in $\mathbb{Q}[x]$.

   (d) If both (i) and (ii) hold then f is irreducible in $\mathbb{Q}[x]$.

   (e) If both (i) and (Iii) hold then f is irreducible in $\mathbb{Q}[x]$.

   (f) If both (ii) and (iii) hold then f is irreducible in $\mathbb{Q}[x]$.

2. Algebra Comprehensive Exam Spring 2019: #9a
   True or False: The polynomial $x^4 + 1$ is irreducible over $\mathbb{R}$.

3. Algebra Comprehensive Exam Fall 2011: #10
   Prove that the two polynomials $f(x) = x^5 - 5x \pm 1$ are irreducible over $\mathbb{Q}$.

4. Algebra Qualifying Exam Spring 2001: #8
   Show that the polynomial $x^5 - 9x + 2$ is irreducible over $\mathbb{Q}$.
   Hint: Check irreducibility modulo a suitable prime. Depending on what prime you choose, you may need to also check that there are no rational roots.

5. Algebra Qualifying Exam Spring 2010: #4
   Let n be a positive integer. Prove that the polynomial $f(x) = x^{2^n} + 8x + 13$ is irreducible over $\mathbb{Q}$.
   Note: Algebra Qualifying Exam Spring 2007 #10 is very similar to this problem. This is also pretty similar to Algebra Qualifying Exam Winter 2000 #14.

6. Algebra Comprehensive Exam Spring 2016: #5
   For which values of $a \in \mathbb{Z}/5\mathbb{Z}$ is the ring $(\mathbb{Z}/5\mathbb{Z})[x]/(x^3 + ax + 2)$ a field?

7. Algebra Comprehensive Exam Spring 2008: #F7
   Is the polynomial $x^6 - 30x^5 + 6x^4 - 18x^3 + 12x^2 - 6x + 12$ irreducible over $\mathbb{Q}$?
   Explain your answer.

8. Algebra Qualifying Exam Fall 2013: #6a
   Prove that $x^2 + x + 2$ is irreducible modulo 5.

9. Algebra Qualifying Exam Spring 2010: #7
   For each of the following two rings, determine whether or not it is a field:

   (a) $(\mathbb{Z}/2\mathbb{Z})[x]/(x^3 + x + 1)$,

(b) $(\mathbb{Z}/3\mathbb{Z})[x]/(x^3+x+1)$.

Note: This showed up also as Algebra Qualifying Exam Spring 2005 #3 and as Algebra Qualifying Exam Winter 2003 #1.

10. Dummit and Foote Exercise 3 Section 9.5
Let p be an odd prime in $\mathbb{Z}$ and let n be a positive integer. prove that $x^n - p$ is irreducible over $\mathbb{Z}[i]$.
Hint: Use the characterization of irreducible elements in $\mathbb{Z}[i]$ and Eisenstein's Criterion.

R-modules

1. Algebra Comprehensive Exam Spring 2010: #R5
   Let M be a module over the ring $R = \mathbb{Z}/p^n\mathbb{Z}$ and suppose that the number of elements of M is finite. Show that $|M| = p^k$ for some integer $k \geq 0$.

2. Algebra Qualifying Exam Fall 2017: #9e
   True/False: For every integral domain R and every R-module M, the set of torsion elements is a submodule.
   Note: This is Exercise 8a in Section 10.1.

3. Algebra Qualifying Exam Spring 2017: #6
   Let R be a commutative ring with 1 and let M be an R-module. Show that if $M \oplus M$ is a finitely generated R-module, then M is a finitely generated R-module.

4. Dummit and Foote Section 10.2 Exercise 1
   Let R be a ring with 1. Use the submodule criterion to show that kernels and images of R-module homomorphisms are submoodules.

5. Dummit and Foote Section 10.2 Exercise 9
   Let R be a commutative ring with identity and let M be a left R-module. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules.
   Hint: Show that each element of $\mathrm{Hom}_R(R, M)$ is determined by its value on the identity of R.

6. Dummit and Foote Section 10.3 Exercise 4
   An R-module M is called a torsion module if for each $m \in M$ there is a nonzero element $r \in R$ such that $rm = 0$ where r may depend on m. That is, $M = \mathrm{Tor}(M)$ in the notation of Section 10.1 Exercise 8.
   (a) Prove that every finite abelian group is a torsion $\mathbb{Z}$-module.
   (b) Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

7. Dummit and Foote Section 10.3 Exercise 6
   Prove that if M is a finitely generated R-module that is generated by n elements then every quotient of M may be generated by n (or fewer) elements. Deduce that quotients of cyclic modules are cyclic.

8. Dummit and Foote Section 10.3 Exercise 7
   Let N be a submodule of M. Prove that if both M/N and N are finitely generated, so is M.

9. Dummit and Foote Section 10.3 Exercise 9
   An R-module M is called irreducible if $M \neq 0$ and if 0 and M are the only submodules of M.
   (a) Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator.
   (b) Determine all the irreducible $\mathbb{Z}$-modules.

10. Dummit and Foote Section 10.3 Exercise 12
    Let R be a commutative ring and let A, B and M be R-modules. Prove the following isomorphisms of R-modules:
    (a) $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$
    (b) $\mathrm{Hom}_R(M, A \times B) \cong \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B)$.

Statements from Lecture

1. Algebra Comprehensive Exam Spring 2017: #6
   Prove that for every prime number p, the polynomial $x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Q}[x]$ is irreducible.
   Note: This is Example 4 on page 310 in Section 9.4. (This also came up as Algebra Comprehensive Exam Spring 2009 #5 and as Algebra Qualifying Exam Fall 2000 #3b.

2. Algebra Comprehensive Exam Fall 2012: #5
   Let F be a field and $f(x)$ be a polynomial in $F[x]$. Suppose that $R = F[x]/(f)$ is an integral domain. Show that in fact R is a field.
   Note: This is basically asking for Proposition 15 in Section 9.5.

3. Algebra Qualifying Exam Fall 2001: #8
   Prove that any finite subgroup of the multiplicative group K^* of a field K is cyclic.
   Note: This is Proposition 18 in Section 9.5.

4. Algebra Qualifying Exam Spring 1997: #5
   Let R be a commutative ring with identity and let $f(x), g(x) \in R[x]$. Assume that the ideals generated by the coefficients of $f(x)$ and $g(x)$ are both R. Prove that the ideal generated by the coefficients of $f(x)g(x)$ is also R.
   Note: This is basically the statement we proved in lecture about the content of a polynomial being multiplicative.

5. Algebra Qualifying Exam Fall 2015: #7
   Give an example of a module M over a ring R such that M is **not** finitely generated as an R-module. Prove that it is not finitely generated as an R-module.

6. Algebra Qualifying Exam Spring 2015: #2 (part of it)
   Give an example of a UFD that is not a PID.
   Note: A question asking for such an example has come up on many exams.

7. Algebra Qualifying Exam Spring 2014: #9
   Give definitions for each of the following:

   (a) group
   (b) ring
   (c) integral domain
   (d) module
   (e) homomorphism of modules.

   Note: This is very close to Algebra Qualifying Exam Fall 2010 #7.

8. Algebra Comprehensive Exam Spring 2017: #7
   Prove that $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.
   Note: We answered this question in Lecture 19 Video 2.