Math 206B: Algebra
Midterm 2: Things to Know

This document continues one from earlier in the course giving a list of definitions, examples, and results that you should know for Midterm 2 on Friday, 2/26.

Polynomial Rings and R-modules

Definitions

1. The content of a polynomial (Lecture 9)
2. The multiplicity of a root (Section 9.5)
3. Vector space over a field and linear transformation (Lecture 13)
4. R-module. Unital R-module.
5. R-submodule.
6. What it means for an R-module to be annihilated by an ideal I of R.
7. R-module homomorphism. R-module isomorphism. Kernel and image.
8. $\mathrm{Hom}_R(M, N)$
9. The endomorphism ring of an R-module.
10. The submodule generated by a subset $A \subset M$.
11. The sum of submodules $N_1, \ldots, N_k \subset M$.
12. Set of generators for an R-module.
13. Finitely generated R-module. Cyclic R-module.
14. R-linear combination. Spanning set. Linearly independent and Linearly dependent. (Section 2 of Conrad's 'Introductory Notes on Modules' notes)
15. The direct product/external direct sum of R-modules.
16. The internal direct sum of R-modules $R_1, \ldots, R_k$.
17. What it means for an R-module M to be free on a subset $A \subset M$. Basis/set of free generators of M. Free module.
18. Bilinear maps (Lecture 19/Conrad's 'Tensor Products' notes.)

Examples

1. Using Eisentstein's criterion to show that x^4+1 and $x^{p-1}+x^{p-2}+\cdots+x+1$ are irreducible.
2. Examples of R-modules: Vector space over a field. Submodules are subspaces.
3. Affine n-space over F. R-module homomorphisms are linear transformations.
4. Examples of R-modules: Free module of rank n over R.
5. Examples of R-modules: R is a left R module over itself. Left submodules are left ideals.
6. Examples of R-modules: R/I.
7. Examples of R-modules: $\mathbb{Z}$-modules are the same as abelian groups. Submodules are the same as subgroups. $\mathbb{Z}$-module homomorphisms are the same as homomorphisms of abelian groups.
8. Examples of R-modules: $R[x]$.
9. Examples of R-modules: $F[x]$-modules. V a vector space together with a linear transformation $T\colon V \to V$. Examples: T is the zero transformation, identity transformation, shift operator. Submodules are subspaces that are T-stable.
10. The same set can have many different R-module structures.
11. The same set can be an R-module for different rings R. The set of submodules can change.
12. A finite abelian group is a $\mathbb{Z}$-module but we cannot extend this action to make M into a $\mathbb{Q}$-module.
13. There is no scalar multiplication making $\mathbb{Z}/5\mathbb{Z}$ into a $\mathbb{Z}[i]$-module.
14. Example of an abelian group homomorphism that is not a an R-module homomorphism. Example of an R-module homomorphism that is not a ring homomorphism. Example of a ring homomorphism that is not an R-module homomorphism.
15. Submodules of a finitely generated module need not be finitely generated.
16. $R[x]$ is not finitely generated as an R-module.
17. Examples of R-modules: R^∞. The standard basis vectors $e_1, e_2, \ldots$ are not a spanning set for R^∞.
18. Contrasts between finitely generated vector spaces and R-modules: R-module that does not have a basis. Single element that is linearly dependent. Maximal linearly independent set that is not a spanning set. Minimal spanning set that is not linearly independent. Spanning set that does not contain a basis. Linearly independent set that cannot be enlarged to a basis. A submodule of a finitely generated free R-module need not be free. A finitely generated free R-module of rank n can strictly contain a free R-module of the same rank. (Lecture 18/Conrad's notes)
19. $\mathbb{Q}$ is not a free $\mathbb{Z}$-module (Lecture 19)

Theorems

1. If R is a UFD then $R[x]$ is a UFD. (Theorem 7 Section 9.3)
2. Gauss' Lemma (Proposition 5 and Corollary 6 Section 9.3)
3. The content of a polynomial is multiplicative (Lecture 9)
4. Let R be an integral domain. A nonconstant monic $p(x) \in R[x]$ is irreducible iff it cannot be written as a product of two monic polynomials of smaller degree. (Section 9.3 page 306)
5. A polynomial in $F[x]$ has a linear factor if and only if it has a root in F. (Proposition 9 in Section 9.4)
6. The Rational Root Test (Proposition 11 in Section 9.4)
7. Proving a polynomial is irreducible by 'reducing the coefficients modulo P' (Proposition 12 in Section 9.4)
8. Eisenstein's criterion (Proposition 13 in Section 9.4)
9. The maximal ideals of $F[x]$ are the ideals $(f(x))$ where $f(x) \in F[x]$ is irreducible. (Proposition 15 in Section 9.5)
10. Structure of $F[x]/(g(x))$ in terms of a factorization of $g(x)$. (Proposition 16 in Section 9.5)
11. A polynomial of degree d in $F[x]$ has at most d roots in F, even counted with multiplicity. (Proposition 17 in Section 9.5)
12. A finite subgroup of the multiplicative group of a field is cyclic. (Proposition 18 in Section 9.5. Note that we gave two proofs of this fact.)
13. The structure of $\mathbb{Z}/n\mathbb{Z}^*$ for all n. (Corollary 20 in Section 9.5)
14. Hilbert's Basis Theorem (Theorem 21 in Section 9.6. Note that we gave a different proof than the one in the textbook.)
15. If M is an R-module annihilated by I, we can make M into an (R/I)-module. (Example 5 page 338-9 Section 10.1) R-module homomorphisms are automatically (R/I)-module homomorphisms.
16. The submodule criterion (Proposition 1 in Section 10.1)
17. The structure of $\text{Hom}_R(M, N)$ and the endomorphism ring of M (Proposition 2 in Section 10.2)
18. Isomorphism Theorems for Modules (Theorem 4 in Section 10.2)
19. Basics of quotients of R-modules (Proposition 3 in Section 10.2)
20. The recognition theorem for direct products of R-modules (Proposition 5 in Section 10.3)
21. Let R be a commutative ring with 1. M is a finitely generated R-module if and only if M is isomorphic to a quotient of R^n for some $n > 0$. (Lecture 19)