Math 206B: Algebra
Tensor Products Problems

In this document, we give some practice problems from past Algebra Comprehensive and Qualifying Exams that involve tensor products.

In these problems you should use the definitions for rings given in Dummit and Foote. For example, a ring R does not necessarily have an identity, and a ring homomorphism between two rings with identity does not necessarily take the identity of the first ring to the identity of the second ring.

Tensor Products

1. Algebra Qualifying Exam Winter 2021: #6
   Let R be an integral domain and let I be a principal ideal of R. Consider I as an R-module. Let M be the R-module $I \otimes_R I$. Prove that the only torsion element of M is zero.
   Note: See the first part of Exercise 21 of Section 10.5 of Dummit and Foote.

2. Algebra Qualifying Exam Spring 2019: #5
   Classify all finite abelian groups G such that $G \otimes_{\mathbb{Z}} (\mathbb{Z}/9\mathbb{Z}) \cong G$.
   Note: See Example 4.6 of Conrad's 'Tensor Products' notes.

3. Algebra Qualifying Exam Spring 2017: #4
   Let G and H be finite abelian groups, and suppose that the order of G is relatively prime to the order of H. Show that $G \otimes_{\mathbb{Z}} H = 0$.
   Note: See the problem directly below this one plus a statement about tensor products of direct sums (Theorem 17 in Section 10.5 of Dummit and Foote or Theorem 5.4 in Conrad's 'Tensor Products' notes).

4. Algebra Qualifying Exam Fall 2013: #4
   Suppose that m and n are relatively prime positive integers. Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = 0$.
   Note: This is Example 3 on page 369 of Dummit and Foote. This also came up as Algebra Qualifying Exam Fall 2011: #10c.

5. Algebra Qualifying Exam Spring 2012: #9d
   True/False: If R is a commutative ring and M, N are nonzero R-modules, then $M \otimes_R N$ is nonzero.

6. Algebra Qualifying Exam Spring 2016: #8c
   True/False: Let n be a positive integer. Then $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. Explain your answer.
   Note: This is Example 2 on page 363 of Dummit and Foote.

7. Algebra Qualifying Exam Spring 2013: #8
   Suppose that R is a commutative ring with identity and M_1 and M_2 are distinct maximal ideals of R. Show that $R/M_1 \otimes_R R/M_2 = 0$.
   Note: This follows from Exercise 16 of Section 10.5 of Dummit and Foote.

8. Algebra Qualifying Exam Fall 2012: #9e
   If V, W are finite dimensional vector spaces over a field F, what is the dimension of the tensor product $V \otimes_F W$?
   Note: See Example 4.10 of Conrad's 'Tensor Products' notes.

9. Algebra Qualifying Exam Fall 2010: #8
   Suppose A is a finite abelian group, S is a Sylow p-subgroup of A, and p^k is the order of S.

Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to S.
Note: See Example 4.6 of Conrad's 'Tensor Products' notes. This is Exercise 5 of Section 10.5 of Dummit and Foote. This also appeared as Algebra Qualifying Exam Spring 2009 #7.

Below are three additional exercises about tensor products that have appeared on recent qualifying exams. Both of these exercises involve things we are not going to cover in Math 206 this year, but I thought I should include them for completeness. Do not worry if you do not know how to solve them.

1. Algebra Qualifying Exam Fall 2020: #8
Give an example of an injective map of abelian groups $M_1 \to M_2$, and an abelian group N, such that $M_1 \otimes_{\mathbb{Z}} N \to M_2 \otimes_{\mathbb{Z}} N$ is not injective. Justify your example by giving an explanation of why it works.
Note: See Example 3 on page 401 of Section 10.5 of Dummit and Foote.

2. Algebra Qualifying Exam Spring 2020: #6
We say that a finitely generated R-module M is invertible if there exists a finitely generated R-module N such that $M \otimes_R N \cong R$.

   (a) Find all invertible $\mathbb{Z}$-modules.

   (b) Prove that if M_1 and M_2 are both invertible then so is $M_1 \otimes_R M_2$.

   (c) Prove that every invertible R-module is projective.

   Note: For a definition of a projective module, see page 390 of Section 10.5 of Dummit and Foote.

3. Algebra Qualifying Exam Fall 2016: #10a
Either give an example of a non-zero zero divisor in $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ or prove that one does not exist. Either way, give a brief explanation.
Note: This is the example on page 375 of Dummit and Foote. I moved this problem here because we did not talk about R-algebras or the situations in which the tensor product of two R-modules is a ring.