Math 206B: Algebra

Final Exam

Thursday, March 18, 2021.

- You have **2 hours** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

T/F Short Answer	
1 (3 Points)	
2 (3 Points)	
3 (3 Points)	
4 (3 Points)	
5 (5 Points)	
6 (5 Points)	
Total	

Problems	
1 (10 Points)	
2 (8 Points)	
3 (6 Points)	
4 (8 Points)	
5 (10 Points)	
Total	

Problems	
6 (10 Points)	
7 (10 Points)	
8 (8 Points)	
9 (8 Points)	
Total	

True/False and Short Answer

1. True or False: If R is a commutative ring with identity and R has a unique prime ideal then R is a field.
   You only need to answer 'True' or 'False'. No other explanation is necessary.

2. True or False: Let R be a PID, M be a finitely generated free R-module, and N be a submodule of M. Then N is free.
   You only need to answer 'True' or 'False'. No other explanation is necessary.

3. True or False: Let R be an integral domain, M be a finitely generated R-module and N be a submodule of M. Then N is finitely generated.
   You only need to answer 'True' or 'False'. No other explanation is necessary.

4. True or False: Let V be a vector space and $V = A \oplus B = C \oplus D$ with $A \cong C$. Then $B \cong D$.
   You only need to answer 'True' or 'False'. No other explanation is necessary.

5. Is there an example of a UFD that is not a PID?
   Either give an example (you do not need to explain why it works), or give a brief explanation for why no such example exists.

6. Let $\mathbb{F}_3$ be a finite field of order 3. Let V be a 3-dimensional vector space over $\mathbb{F}_3$.
   How many 2-dimensional subspaces are contained in V?
   You only need to write down a number. No other explanation is necessary.

1 Problems

1. Let V, U, and W be finite dimensional vector spaces over $\mathbb{C}$. Suppose that $\phi\colon V \to U$ is an injective linear transformation and $\psi\colon U \to W$ is a surjective linear transformation.
   Suppose that $\psi \circ \phi = 0$ and that $\dim U = \dim V + \dim W$.
   Prove that $\ker(\psi) = \operatorname{Im}(\phi)$ as subspaces of U.

2. Let R be a PID and let M be a finitely generated R-module.
   Describe the structure of $M/\operatorname{Tor}(M)$.

3. Let R be a ring and let M be a left R-module. Let

   $$M_1 \subseteq M_2 \subseteq \cdots$$

   be a chain of submodules of M. Let

   $$N = \bigcup_{i=1}^{\infty} M_i.$$

   Prove that N is a submodule of M.

4. Let R be a commutative ring with 1 and M be any R-module. **Prove** that $R \otimes_R M \cong M$.

5. Suppose A is a finite abelian group, S is a Sylow p-subgroup of A, and p^k is the order of S.
   Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes_{\mathbb{Z}} A$ is isomorphic to S.

6. For which values of $a \in \mathbb{Z}/5\mathbb{Z}$ is the ring $(\mathbb{Z}/5\mathbb{Z})[x]/(x^3 + ax + 2)$ a field?
   Prove that your answer is correct.

7. **Prove** that a finite subgroup of the multiplicative group of a field is cyclic.

8. Find the greatest common divisor $d(X)$ of the polynomials

   $$f(X) = X^4 - X^2 + 2X - 1, \quad \text{and } g(X) = X^4 + 2X^3 + X^2 - 1$$

   in $\mathbb{R}[X]$.

9. **Prove** that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.