Math 206B: Algebra

Midterm 2

Friday, February 26, 2021.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (8 Points)	
2 (12 Points)	
3 (3 Points)	
4 (10 Points)	
5 (8 Points)	
Total	

Problems	
6 (5 Points)	
7 (10 Points)	
8 (8 Points)	
9 (12 Points)	
Total	

Problems

1. Let F be a field and $f(x) \in F[x]$.
   Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

2. (a) Let R be a ring with a 1. Give the definition of a left R-module.
   (b) Define what it means for a left R-module M to be free on a subset $A \subseteq M$.
   (c) Let M and N be R-modules.
   Define what it means for a map $\varphi : M \to N$ to be an R-module homomorphism.
   (d) Suppose M and N are both R-modules and that both M and N are rings.
   Give an example of a map $\varphi : M \to N$ that is an R-module homomorphism but not a ring homomorphism.
   Explain why your example works.

3. **State whether the following claim is true or false. No Explanation is Necessary.**

   Suppose R is an *integral domain*.
   If $f(x) \in R[x]$ has degree d, then $f(x)$ has at most d distinct roots in R.

4. All of the following are isomorphic as $\mathbb{R}$-vector spaces, but only two of the following are isomorphic as rings. Which two?
   Explain why they are isomorphic as rings.

   (a) $\mathbb{C} \times \mathbb{C}$
   (b) $\mathbb{C}[x]/(x^2)$
   (c) $\mathbb{C}[x]/(x^2 + 1)$
   (d) $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$
   (e) $\mathbb{R}[x]/(x^4)$

5. What are all of the maximal ideals in the ring $\mathbb{Q}[x]/(x^3 + x^2)$?
   Explain how you know that this is a complete list.

6. **Prove** that the polynomial $x^4 + 15x^3 + 20x^2 + 10x + 45$ is irreducible over $\mathbb{Q}$.

7. For which primes p is the quotient $(\mathbb{Z}/p\mathbb{Z})[x]/(x^2 + x + 1)$ a field?
   Prove that your answer is correct.

8. Let $G = \mathbb{Z}/25\mathbb{Z}$ the cyclic **group** of order 25.
   Can G be given the structure of a (unital) $\mathbb{Z}/5\mathbb{Z}$-module?
   Explain your answer.

9. (a) Is there a ring R with identity and an R-module M such that M is torsion-free and no linearly independent subset generates M?

   (b) Is there a ring R with identity and an R-module M such that M is free, $A \subseteq M$ is a maximal linearly independent set, but A does not generate M?

   For each part either give an example and **prove** that it satisfies the property you are claiming, or **prove** that no such example exists.