Math 206C: Algebra
Homework 4

Due Monday, May 10th at 11:59PM.
Please email `nckaplan@math.uci.edu` with questions.

1. Algebra Comprehensive Exam Fall 2011 #8
   Let F be a field and $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$. Let K_f be a splitting field for $f(x)$ over F. Prove that $[K_f : F]$ divides $n!$.
   Note: We discussed this problem in Lecture 12 but did not give all of the details.

2. Algebra Comprehensive Exam Spring 2013 #9
   Show that $f(x) = x^4 - 2$ and $g(x) = x^4 + 2$ have the same splitting field over $\mathbb{Q}$. Denote this splitting field by K. Find $[K : \mathbb{Q}]$ and give a basis for K over $\mathbb{Q}$.
   Note: We discussed this problem in Lecture 12 but did not give all of the details.

3. Exercise 5 in Section 13.4.
   When discussing what it means for a field to be algebraically closed, we talked about the difference between every polynomial in $K[x]$ splitting completely in $K[x]$ and every polynomial in $K[x]$ having a root in K. In this exercise you consider this kind of issue for splitting fields and irreducible polynomials.

4. Exercise 1 in Section 13.5. Also prove that

   $$D_x((x-\alpha)^n) = n(x-\alpha)^{n-1}.$$

   In this exercise you prove some basic properties of the derivative that we used in Lecture 14.

5. Exercise 5 in Section 13.5
   In this exercise you show that every element of a certain family of polynomials in $\mathbb{F}_p[x]$ is irreducible and separable over $\mathbb{F}_p$.

6. Exercise 6 in Section 13.5
   In this exercise you proof a result that implies *Wilson's theorem* in number theory.
   Note: Wilson's theorem came up as Algebra Comprehensive Exam Spring 2009 #3.

7. Exercise 7 in Section 13.5.
   We mentioned this exercise at the end of Lecture 14. In lecture we proved that if K is a field of characteristic p that is perfect, every irreducible polynomial in $K[x]$ is separable. In this exercise you consider what happens for fields of characteristic p that are **not perfect**.

8. Exercise 11 in Section 13.5
   This exercise is closely related to one of the results from Conard's 'Separability' notes that I stated in Lecture 14, but did not prove.

9. Algebra Qualifying Exam Fall 2020 #5
   Suppose that K is a field of characteristic 5. For which values of $n > 1$ is the polynomial $f(x) = x^n - x$ separable?

10. Algebra Qualifying Exam Spring 2016 #5
    Let K be a field and $\overline{K}$ be an algebraic closure of K. Assume $\alpha, \beta \in \overline{K}$ have degree 2 and 3 over K, respectively.

    (a) Can $\alpha\beta$ have degree 5 over K? Either give an example or prove that this is impossible.

(b) Can $\alpha\beta$ have degree 6 over K? Either give an example or prove that this is impossible.

11. Algebra Qualifying Exam Spring 2017 #8
Suppose F is a **perfect** field and $f(x) \in F[x]$ is a nonconstant polynomial. Show that $F[x]/((f(x))$ is a direct product of fields if and only if $f(x)$ is a separable polynomial.
Note: An earlier version of this problem did not require the assumption that F is perfect. It really is necessary– the problem is not correct as stated if F is not perfect (see Exercise #7).