Math 206C: Algebra
Midterm 1 Practice Problems

The goal of this document is to provide you with some practice problems for Midterm 1 from past Algebra Comprehensive and Qualifying Exams. I have made an attempt to divide up the problems by topic and also to indicate which ones we have already proven in lecture.

Describe all Matrices Satisfying Some Equation

1. Algebra Advisory Exam Fall 2005 #L10
   Describe all matrices in $A \in \mathrm{Mat}_5(\mathbb{C})$ satisfying $A^2 - A = 0$.
2. Algebra Comprehensive Exam Spring 2005 #L2
   Describe all matrices in $A \in \mathrm{Mat}_4(\mathbb{R})$ satisfying $A^4 = 0$.
3. Algebra Comprehensive Exam Spring 2007 #L2
   Find matrices in $\mathrm{Mat}_3(\mathbb{C})$ that satisfy the equation $X^3 = X$.
   Note: This also came up as Algebra Comprehensive Exam Fall 2004 #7.
4. Algebra Advisory Exam Fall 2010 #L6
   Find all $A \in \mathrm{Mat}_4(\mathbb{R})$ such that $A^3 = I$, where I is the identity matrix.
5. Algebra Comprehensive Exam Spring 2008 #L9
   Find all matrices in $X \in \mathrm{Mat}_3(\mathbb{C})$ such that $X^2 - X = 0$ up to similarity.
   Use the Jordan canonical form.
6. Algebra Qualifying Exam Winter 2003
   Let p be a prime, V be a vector space of dimension p over $\mathbb{Q}$, and $T\colon V \to V$ be a linear transformation such that $T^p = I$. Find all possible rational canonical forms for T and the characteristic polynomial of each.

Characteristic and Minimal Polynomials/Invariant Factors

1. Algebra Qualifying Exam Spring 2014 #8
   Prove that two 3×3 matrices over a field are similar if and only if they have the same characteristic and minimal polynomial.
   Note: We proved this in Lecture 3. It is part of Exercise 4 of Section 12.2.
2. Algebra Qualifying Exam Fall 2010 #10
   (a) How many similarity classes of matrices over $\mathbb{Q}$ have characteristic polynomial $(x^4 - 1)(x^2 - 1)$?
   (b) Find one representative from each similarity class.
   (c) For each similarity class, give the minimal polynomial of the matrices in that class.

   Note: We solved this problem in lecture. It is Example (4) on pages 486-487 in Section 12.2.

3. Algebra Qualifying Exam Fall 2014 #9
   Determine all real matrices A with characteristic polynomial $x^3(x^2+1)$, up to conjugation.
   Note: Algebra Qualifying Exam Spring 2010 #10 is extremely similar to this question.

4. Algebra Qualifying Exam Spring 2005 #6
   Let M be a matrix over $\mathbb{Q}$ with characteristic polynomial $(x+1)^2x^4$ and minimal polynomial $(x+1)^2x^2$.

   (a) Find $\mathrm{tr}(M)$ and $\det(M)$.

   (b) How many distinct conjugacy classes of such matrices are there in $\mathrm{GL}_6(\mathbb{Q})$? Explain.
   Note: I think this is phrased in a potentially confusing way. –Nathan

   (c) Write down a 6×6 matrix with entries in $\mathbb{Q}$ having the above characteristic and minimal polynomials.

5. Algebra Qualifying Exam Fall 2008 #10
   Let A be a square matrix with entries in $\mathbb{Q}$ and with characteristic polynomial $(x^3+2)^2(x^2-3)$.

   (a) What are the possibilities for the minimal polynomial of A?

   (b) What are the trace and determinant of A?

   (c) How many distinct conjugacy classes of such matrices are there in $\mathrm{GL}_8(\mathbb{Q})$? For each conjugacy class, give one matrix A in that conjugacy class.

6. Algebra Qualifying Exam Fall 2004 #2
   Let M be a 9×9 matrix over $\mathbb{C}$ with characteristic polynomial $(x^2+1)^3(x+1)^3$ and minimal polynomial $(x^2+1)(x+1)$.

   (a) Find $\mathrm{tr}(M)$ and $\det(M)$.

   (b) How many distinct conjugacy classes of matrices are there in $\mathrm{GL}_9(\mathbb{C})$. Explain.

   (c) Write down a 9×9 matrix with coefficients in $\mathbb{Q}$ having the above characteristic and minimal polynomials.

7. Algebra Qualifying Exam Spring 2012 #8
   Which of the following matrices are similar over $\mathbb{Q}$? Which are similar over $\mathbb{C}$?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

8. Algebra Qualifying Exam Spring 2010 #8
   List exactly one representative from each similarity class of matrices $A \in \mathrm{GL}_2(\mathbb{C})$ such that A is similar to A^{-1}.
   Hint: How are the eigenvalues of A related to the eigenvalues of A^{-1}?

9. Algebra Advisory Exam Fall 2009 #4

   (a) Give an example of a square matrix A with entries in $\mathbb{Q}$ that has characteristic polynomial $X^5 - X^3$ and minimal polynomial $X^4 - X^2$.

   (b) Suppose A and B are square matrices with coefficients in a field F and both A and B have characteristic polynomial $X^5 - X^3$ and minimal polynomial $X^4 - X^2$. Show that A and B are similar.

10. Algebra Qualifying Exam Fall 2020 #9
For a matrix $A \in \mathrm{Mat}_n(\mathbb{R})$, prove that the following are equivalent:

    (a) the only eigenvalue of A is $\lambda = 0$;
    (b) there exists $m \geq 1$ such that A^m is the zero matrix;
    (c) A^n is the zero matrix.

11. Algebra Qualifying Exam Spring 2010 #5
Let T be a linear operator on an n-dimensional vector space V over a field F.
Assume that T is nilpotent. Show that $T^n = 0$.
Note: This also came up as Algebra Qualifying Exam Spring Fall 2007 #7.

12. Algebra Qualifying Exam Spring 2018 #7
Let K be a field and let A be an $n \times n$ matrix over K. Suppose that $f \in K[x]$ is an irreducible polynomial such that $f(A) = 0$. Show that $\deg(f) \mid n$.

13. Algebra Qualifying Exam Fall 2009 #6
Let $p > 2$ be prime. Let V be a finite dimensional vector space over $\mathbb{Q}$ of dimension not divisible by $p-1$ and $T: V \to V$ be a linear transformation.
Show that $T^{p-1} + \cdots + T + I \neq 0$, where I is the identity map on V.

14. Algebra Qualifying Exam Fall 2012 #6
How many conjugacy classes are there in the group $\mathrm{GL}_3(\mathbb{F}_2)$?
Hint: Use rational canonical forms and/or the fundamental theorem of finitely generated modules over $\mathbb{F}_2[x]$.

15. Algebra Advisory Exam Fall 2006 #L9
Let V be a vector space and $T: V \to V$ be a linear transformation. If $T^3 = T^2$ does it follow that $T^2 = T$? Justify your answer.

16. Algebra Qualifying Exam Fall 2019 #9
Let V be a finite dimensional vector space over $\mathbb{F}_2$ and let $T: V \to V$ be a linear transformation such that $T^5 = I$ but $T \neq I$.

    (a) What are the possible degrees of the minimal polynomial of T?
    (b) Assume that there are non-similar linear transformations $S, T: V \to V$ such that $S^5 = I = T^5$ but $S \neq I \neq T$. What is the smallest possible dimension of V?

17. Algebra Advisory Exam Fall 2008 #L9
Let V be a finite dimensional vector space and $T: V \to V$ be a linear transformation.
Suppose that the minimal polynomial of T is $p(x) = x^3 - 2x^2 - x + 2$.
Prove that T is diagonalizable.

18. Algebra Comprehensive Exam Spring 2010 #L8
Let V be a finite dimensional vector space and $T: V \to V$ be a linear transformation.

    (a) Show that if $v_1 \ldots, v_k$ are eigenvectors of T corresponding to k distinct eigenvalues $c_1, \ldots, c_k$, then thee v_i are linearly independent.
    Hint: Argue by induction. Suppose that $v_1, \ldots, v_{j-1}$ are linearly independent but that v_j is a linear combination of them and derive a contradiction.
    (b) Deduce that if V is n-dimensional and if the characteristic polynomial of T has n distinct roots in F, then T is diagonalizable.

19. Exercise 37 of Section 12.3
In this exercise you consider the Jordan canonical form of J^2, where J is a Jordan block of size n with eigenvalue λ.

20. Algebra Qualifying Exam Winter 2021 #9
Assume $A \in \mathrm{Mat}_n(\mathbb{C})$ is a matrix such that all eigenvalues of A are non-zero. Prove that A has a square root in $\mathrm{Mat}_n(\mathbb{C})$, that is, there is a matrix $B \in \mathrm{Mat}_n(\mathbb{C})$ such that $A = B^2$.
Hint: It may be helpful to examine the Jordan canonical form of the square of a Jordan cell. (That is, do Exercise 37 of Section 12.3 first.)
Note: This is basically Exercise 38 of Section 12.3.

21. Algebra Qualifying Exam Spring 2011 #8
Suppose that A is an $n \times n$ matrix with entries in $\mathbb{C}$ with minimal polynomial $(x - \lambda)^n$.
    (a) What is the Jordan form of A?
    (b) What is the Jordan form of A^2 when $\lambda \neq 0$?
    (c) What is the Jordan form of A^2 when $\lambda = 0$?

22. Exercise 26 of Section 12.3
In this exercise you compute the Jordan canonical form of the $n \times n$ matrix with entries in $\mathbb{F}_p$ that has every entry equal to 1.

23. Algebra Qualifying Exam Winter 2021 #10
Find a non-singular matrix $A \in \mathrm{Mat}_n(\mathbb{F}_5)$ of smallest possible dimension n such that $A^2 + 2I$ is its own inverse and A is not a scalar multiple of I.

24. Algebra Qualifying Exam Winter 2000 #2
Let T be a linear operator of an n-dimensional vector space V over a field F (not necessarily algebraically closed), where n is a positive integer. Show that there is a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$ has at least $n(n-1)/2$ zero entries.

Invariant Factors and Dimension

In these questions you are asked to prove something about the dimension of an $F[x]$-module or the size of a certain matrix.

1. Algebra Qualifying Exam Spring Fall 2007 #6
   Let V be a finite dimensional vector space of dimension n over $\mathbb{Q}$. Let $A \in \mathrm{End}_{\mathbb{Q}}(V)$ be a linear map from V to itself. Assume that $A^5 = I$. Assume further that if $v \in V$ is such that $Av = v$ then $v = 0$. Show that the dimension of n is divisible by 4.

2. Algebra Qualifying Exam Spring 2016 #7
   Let $T \colon V \to V$ be a linear transformation of a vector space V over $\mathbb{Q}$ that satisfies the relation $(T^3 + 3I)(T^3 - 2I) = 0$. Show that the dimension of V is divisible by 3.

3. Algebra Qualifying Exam Fall 2015 #3
   Let A be an $n \times n$ matrix with entries in $\mathbb{R}$ such that $A^2 = -I$.

   (a) Prove that n is even.

   (b) Prove that A is diagonalizable over $\mathbb{C}$ and describe the corresponding diagonal matrices.

4. Algebra Qualifying Exam Spring 2013 #7
   Let V be a finite dimensional vector space of dimension n over $\mathbb{Q}$. Let $A \in \mathrm{End}_{\mathbb{Q}}(V)$ be a linear transformation from V to itself. Assume that $A^7 = I$. Assume further that A has no non-zero fixed points in V. Show that the dimension of V is divisible by 6.

Eigenvectors and Eigenvalues

This question focuses on eigenvalues and does not fit so nicely into one of the categories above.

1. Algebra Comprehensive Exam Spring 2009 #7
   Let V be a finite dimensional vector space over $\mathbb{C}$ and $T \colon V \to V$ be a linear transformation. Suppose $p(x) \in \mathbb{Q}[x]$.

   (a) Define $p(T)$.

   (b) Show that if λ is an eigenvalue of T, then $p(\lambda)$ is an eigenvalue of $p(T)$.

   (c) Show that if λ is an eigenvalue of $p(T)$, then there is an eigenvalue λ' of T such that $\lambda = p(\lambda')$.

   Note: I would not ask you something like part(c) on Midterm 1. I have seen a few solutions and they all involve the Fundamental Theorem of Algebra (which is proven in Section 14.6).

Additional Exercises

These exercises are not the kind of thing you should expect to see on an exam, but they highlight different perspectives from which you could view some of the material we have developed in the early part of this course.

1. Exercises 29 and 30 of Section 12.3
   Generalized eigenspaces are defined in the paragraph above Exercise 29. In lecture we described how to compute the Jordan canonical form of a matrix A by diagonalizing the matrix $xI - A$. These two exercises describe an alternate way to compute the Jordan canonical form by computing the ranks of certain linear transformations.

2. Algebra Qualifying Exam Spring 2019 #10
   The *nilpotency index* of a nilpotent matrix X is the smallest positive integer k such that $X^k = 0$.

   Suppose $A, B \in \mathrm{Mat}_n(\mathbb{C})$ satisfy the following:

   (a) A and B are nilpotent with the same nilpotency index.
   (b) $\mathrm{rank}(A) = \mathrm{rank}(B)$.
   (c) $\mathrm{rank}(A^2) = \mathrm{rank}(B^2)$.

   Prove the following:

   (a) If $n \geq 9$ then A, B may not be similar.
   (b) If $n \leq 9$ then A and B are similar.

   Note: The reason this problem is here and not in the section above is that I think it is helpful to think about the connection between generalized eigenspaces, Jordan blocks, and the ranks of certain matrices.

3. Algebra Qualifying Exam Fall 2011 #9
   Suppose A is an 8×8 matrix with with entries in $\mathbb{C}$ such that:

   - $\dim_{\mathbb{C}}(\ker(A - 2I)) = 2$,
   - $\dim_{\mathbb{C}}(\ker(A - 2I)^2) = 3$,
   - $\dim_{\mathbb{C}}(\ker(A - 3I)) = 2$,
   - $\dim_{\mathbb{C}}(\ker(A - 3I)^2) = 4$,
   - $\dim_{\mathbb{C}}(\ker(A - 3I)^3) = 5$.

   (a) What is the characteristic polynomial of A?
   (b) What is the minimal polynomial of A?
   (c) What is the Jordan normal form of A?
   (d) What is the rational canonical form of A?

   Note: Like the problem above, I think the best way to do this problem is to use the ideas of Exercises 29 and 30 of Section 12.3.

4. Algebra Qualifying Exam Fall 2006 #10
   Suppose that p is a prime and M is an $\mathbb{F}_p[x]$-module. Suppose that $(x^3 - 1)M = 0$ and $|(x-1)^2M)| = p$ and $|(x-1)M| = p^3$ and $|M| = p^7$. Determine M as an $\mathbb{F}_p[x]$-module, up to isomorphism.
   Note: This problem was originally in a different section, but I think the best way to think about it is to think about the information in this problem in terms of the dimension of the kernel of $(T - I)$ and of $(T - I)^2$.

5. Exercise 35 of Section 12.3
   This exercise describes an interpretation of the diagonal entries of the Smith normal form of A in terms of the gcd's of collections of $i \times i$ minors of A.

6. Algebra Qualifying Exam Fall 2013 #4
   Let M be an $n \times n$ matrix over a field F.

   (a) If F has characteristic 0 show that M is nilpotent if and only if $\mathrm{tr}(M^i) = 0$ for all $1 \leq i \leq n$.

   (b) Give an example to show that the same statement is not true if the field F has positive characteristic $p > 0$.

   Note: I think that direction of the first part of this problem is pretty tricky.