Math 206C: Algebra
Midterm 2- Fields Practice Problems

The goal of this document is to provide you with some practice problems for Midterm 2 from past Algebra Comprehensive and Qualifying Exams. I have made an attempt to divide up the problems by topic and also to indicate which ones we have already proven in lecture. In this document we will focus on problems that we can solve without using anything from our unit on Galois theory.

Canonical Forms and Matrices

1. Algebra Qualifying Exam Fall 2016 #7
   Give an example of a 10×10 matrix over $\mathbb{R}$ with minimal polynomial $(x^4 - 2)(x + 2)^2$ which is not similar to any matrix with rational entries. Briefly explain your answer.
   Note: The reason I did not post this problem before Midterm 1 is that you might want to think about the roots of $x^4 - 2$ in $\mathbb{R}$. We considered this example when discussing splitting fields.

2. Algebra Comprehensive Exam Spring 2019 #7
   Suppose A is an $n \times n$ matrix over a field K with minimal polynomial $m(x)$. Let $f(x) \in K[x]$ be a polynomial. Prove that $f(A)$ is nonsingular if and only if $f(x)$ and $m(x)$ are relatively prime in $K[x]$.

3. Algebra Qualifying Exam Fall 2004 #10
   (a) Find all positive integers which occur as the order of some element of $\mathrm{GL}_2(\mathbb{Q})$. Exhibit an element of $\mathrm{GL}_2(\mathbb{Q})$ of order 3.
   (b) Find all positive integers which occur as the order of some element of $\mathrm{GL}_2(\mathbb{R})$. Exhibit an element of $\mathrm{GL}_2(\mathbb{R})$ of order 11.

   Note: This second part involves something we have not talked about in lecture in Math 206.

4. Let V be a finite dimensional vector space over $\mathbb{Q}$ and $T \colon V \to V$ be a linear transformation satisfying $T^{15} = I$. Assume that both T^3 and T^5 have no non-zero fixed points in V. Show that the dimension of V is divisible by 8.
   Note: The reason I did not post this problem before Midterm 1 is that you will probably want to factor the polynomial $T^{15} - 1$, which we didn't necessarily know how to factor earlier this quarter.

Finite Fields

1. Algebra Comprehensive Exam Spring 2020 #10
   Let k be a finite field and let $\sigma \colon k \to k$ be defined by $\sigma(x) = x^2$.
   (a) Assume the cardinality of k is even. Is σ surjective? Briefly explain.
   (b) Assume the cardinality of k is odd. Is σ surjective? Briefly explain.

2. Algebra Comprehensive Exam Spring 2015 #10
   Construct a field with 32 elements. Prove that your construction produces a field with exactly 32 elements.

3. Algebra Qualifying Exam Winter 2021 #7
   Find a polynomial $p(x) \in \mathbb{F}_2[x]$ such that $\mathbb{F}_2[x]/(p(x))$ is a finite field with 8 elements. Prove that your polynomial works.

4. Algebra Qualifying Exam Fall 2001 #10
   (a) Find an irreducible polynomial $f(x) \in \mathbb{F}_2(x)$ of degree 3. Explain why $\mathbb{F}_8 \cong \mathbb{F}_2(\alpha)$ where α is any root of $f(x)$.
   (b) Let $B = \{1, \alpha, \alpha^2\}$. This is a basis for $\mathbb{F}_8$ over $\mathbb{F}_2$. Let $T_2 \colon \mathbb{F}_8 \to \mathbb{F}_8$ be the linear map defined by $T_2(\beta) = \beta^2$. Compute the matrix of T_2 relative to the basis B.

Separable Polynomials and Separable Extensions

1. Algebra Qualifying Exam Fall 2011 #10d
   True or False: If F is a finite extension of $\mathbb{Q}$ in $\mathbb{C}$, and $F \not\subset \mathbb{R}$, then $[F \colon \mathbb{R}]$ must be even.

2. Algebra Qualifying Exam Fall 2009 #9
   Suppose that F is an algebraically closed field. Find all monic separable polynomials $f(x) \in F[x]$ such that the set of zeros of $f(x)$ in F is closed under multiplication.
   Note: This also came up as Algebra Qualifying Exam Spring 2008 #10 and as Algebra Qualifying Exam Spring 1996 #10.

3. Algebra Qualifying Exam Fall 2012 #10(a,c)
   For each of the following, either give an example or explain briefly why no such example exists:
   (a) A quadratic extension of fields that is not separable.
   (b) An infinite field where every nonzero element has finite multiplicative order.

4. Algebra Qualifying Exam Spring 2005 #7
   Suppose p is a prime number and L/K is a field extension of degree p.
   (a) Prove that if $K = \mathbb{Q}$, then L/K is separable.
   (b) Prove that if $K = \mathbb{F}_p$, then L/K is separable.
   (c) Give an example of a field extension L/K of degree p that is **not** separable.

5. Algebra Qualifying Exam Spring 2015 #9
   (a) What does it mean for a field to be **perfect**?
   (b) Give an example of a perfect field. (No need to justify your answer.)
   (c) Give an example of a field that is **not** perfect. (No need to justify your answer.)

Cyclotomic Polynomials

1. Algebra Qualifying Exam Spring 1997 #8
   Find the minimal polynomial of a primitive complex 20^{th} root of unity over $\mathbb{Q}$.

Problems from Lecture and Homework

1. Algebra Qualifying Exam Fall 2019 #6
Assume that $F \le K \le L$ are field extensions such that both K/F and L/K are algebraic. Prove that L/F is algebraic.
Note: We proved this in Lecture 11. It is Theorem 20 in Section 13.2.

2. Algebra Advisory Exam Fall 2010 #F9
Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the subfield of elements that are algebraic over $\mathbb{Q}$. Prove that $\overline{\mathbb{Q}}$ is an extension of $\mathbb{Q}$ of infinite degree.
Note: We proved this in Lecture 11. See Example 1 on page 527 in Section 13.2.

3. Algebra Advisory Exam Fall 2008 #F9
Let E and K be extensions of the field F. Suppose $[E\colon F] = n$ and $[K\colon F] = m$ where m and n are relatively prime. Prove that $[EK\colon F] = mn$.
Note: We proved this in Lecture 11. See Corollary 22 in Section 13.2.

4. Algebra Comprehensive Exam Spring 2011 #8
Let f be a polynomial of degree $n > 0$ over a field F. Let K_f be a splitting field for f over F, that is, K_f is obtained by adjoining all the roots of f in an algebraic closure of F. Show that the extension degree $[K_f\colon F]$ divides $n!$.
Hint: Use induction on n.
Note: We discussed this problem in Lecture 12. It was on HW4.

5. Algebra Comprehensive Exam Spring 2013 #9
Show that the polynomials
$$f(x) = x^4 - 2 \quad \text{and} \quad g(x) = x^4 + 2$$
have the same splitting field over $\mathbb{Q}$. (Denote this splitting field by K.) Find the degree of the extension $K/\mathbb{Q}$, and a basis for K over $\mathbb{Q}$.
Note: We discussed this problem in Lecture 12. It was on HW4.

6. Algebra Comprehensive Exam Fall 2020 #7
Assume $f(x) \in \mathbb{Q}[x]$ is an irreducible degree 5 polynomial, and assume there exists $\alpha \in \mathbb{C}$ such that both α and $-\alpha$ are roots of $f(x)$. Prove that no splitting field of $f(x)$ over $\mathbb{Q}$ can contain a primitive 5^{th} root of unity.
Note: We discussed this problem in Lecture 12.

7. Algebra Qualifying Exam Spring 2016 #5
Let K be a field and $\overline{K}$ be an algebraic closure of K. Assume $\alpha, \beta \in \overline{K}$ have degree 2 and 3 over K, respectively.
   (a) Can $\alpha\beta$ have degree 5 over K? Either give an example or prove that this is impossible.
   (b) Can $\alpha\beta$ have degree 6 over K? Either give an example or prove that this is impossible.

   Note: This was a problem on HW4.

8. Algebra Comprehensive Exam Fall 2012 #9
Suppose that $\alpha, \beta \in \mathbb{C}$ have minimal polynomials over $\mathbb{Q}$ of degree 2 and 3 respectively. Can $\alpha + \beta$ have minimal polynomial over $\mathbb{Q}$ of degree 5?
Give an example or prove that it is not possible.
Note: This problem is very similar to the previous one.

9. Algebra Comprehensive Exam Spring 2006 #F9
   Let F be a field. Show that a polynomial $f(x) \in F[x]$ has no multiple roots if and only if $f(x)$ and its derivative $f'(x)$ are relatively prime.
   Hint: You may assume that f has a splitting field E containing F.
   Note: We proved this in Lecture 14. It is Proposition 33 in Section 13.5.

10. Algebra Qualifying Exam Spring 2014 #10
    Prove that every finite field is perfect, i.e., that every finite extension of a finite field is separable.
    Note: This is not quite the definition that Dummit and Foote use for what it means for a field of characteristic p to be perfect. We proved this statement in Lecture 14. See Corollaries 36 and 39 of Section 13.5. This also came up as Algebra Qualifying Exam Fall 2007 #2.

11. Algebra Comprehensive Exam Spring 2009 #3
    Let $p > 2$ be prime. Prove that $1 \cdot 2 \cdots (p-1) \equiv -1 \pmod{p}$.
    Note: This is called Wilson's Theorem. It is Exercise 6 in Section 13.5. This was a problem on HW4.

12. Algebra Qualifying Exam Spring 2017 #8
    Suppose F is a **perfect** field and $f(x) \in F[x]$ is a nonconstant polynomial. Show that $F[x]/((f(x))$ is a direct product of fields if and only if $f(x)$ is a separable polynomial.
    Note: This was a problem on HW4.

13. Algebra Qualifying Exam Fall 2014 #10
    Suppose p is prime and r and N are positive integers. Consider the map $\sigma\colon \mathbb{F}_{p^r}^* \to \mathbb{F}_{p^r}^*$ defined by $\sigma(x) = x^N$. What is the cardinality of its kernel and image? Fully justify.
    Note: This was a question on HW 5.

14. Algebra Comprehensive Exam Spring 2017 #5
    Let p, q be primes and e, f be positive integers. Let $\mathbb{F}$ be a finite field of cardinality q^f. Suppose that $\mathbb{F}$ has a subfield of cardinality p^e. Prove that $p = q$ and that e divides f.
    Note: This follows from a problem on HW5, but we will also prove it later. See page 588 in Section 14.3.

15. Algebra Qualifying Exam Fall 2011 #2
    Let p be a prime and F be an algebraically closed field of characteristic p. Let $n = p^a m$ where $p \nmid m$. How many n^{th} roots of unity are there in F? Prove your answer.
    Note: This was a problem on HW5.

16. Algebra Qualifying Exam Spring 2020 #9
    Let p be a prime and L be a finite field with p^2 elements. Consider the map $\varphi\colon L \to L$ defined by $\varphi(a) = a^p$. If L is viewed as a vector space over $\mathbb{F}_p$, show that φ is $\mathbb{F}_p$-linear and find its characteristic and minimal polynomials.
    Note: This problem follows from some exercises we solved on HW5.