Math 206C: Algebra
Midterm 2 Galois Theory Practice Problems

The goal of this document is to provide some practice problems for Midterm 2 from past Algebra Comprehensive and Qualifying Exams. In this document we will focus on problems that involve ideas from our discussion of Galois theory (starting with Lecture 16).

1. Algebra Qualifying Exam Spring 2012 #9e
   Are the fields $\mathbb{Q}(\sqrt{7})$ and $\mathbb{Q}(\sqrt{11})$ isomorphic (as fields)? Explain.
   Note: Compare this to Exercise 4 in Section 14.1.

2. Algebra Comprehensive Exam Spring 2005 #F2
   Let $F < E < \overline{F}$ be field extensions where $\overline{F}$ is an algebraic closure of F and suppose that E is a splitting field for a family $\mathcal{F}$ of polynomials over F.
   Prove that any embedding σ of E into $\overline{F}$ over F is an automorphism of E.
   Note: This is closely related to part of the proof of Theorem 14 in Section 14.2.

3. Algebra Qualifying Exam Spring 2019 #8
   Let K/F be a Galois algebraic extension with no proper intermediate fields.
   Prove that $[K\colon F]$ is prime.
   Note: This also came up as Algebra Qualifying Exam Fall 2019 #9.

4. Algebra Qualifying Exam Spring 2016 #9b
   Either give an example or state that none exists. In either case, give a **brief** explanation.
   A tower of field extensions $L \supseteq K' \supseteq K$ such that L/K' and K'/K are Galois extensions but L/K is not Galois.
   Note: This is a question we discussed in lecture.

5. Algebra Qualifying Exam Fall 2012 #9b
   True or False: If K/F is a Galois extension with cyclic Galois group, and E is a field satisfying $F \subseteq E \subseteq K$, then E/F is also Galois with cyclic Galois group.

6. Exercise 3 in Section 14.2
   Determine the Galois group of $(x^2-2)(x^2-3)(x^2-5)$. Determine all the subfields of the splitting field of this polynomial.

7. Algebra Qualifying Exam Fall 2006 #2
   Let L be the splitting field of x^3-2 over $\mathbb{Q}$.

   (a) Find $[L\colon \mathbb{Q}]$.
   (b) Describe the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ both as an abstract group and as a set of automorphisms.

   Note: This is an example we have discussed in lecture.

8. Algebra Qualifying Exam Fall 2015 #5
   Construct a Galois extension F of $\mathbb{Q}$ satisfying $\mathrm{Gal}(F/\mathbb{Q}) \cong D_8$, the dihedral group of order 8. Fully justify.
   Note: This also came up as Algebra Qualifying Exam Spring 2007 #1.

9. Algebra Qualifying Exam Spring 2010 #9
   Determine the splitting field over $\mathbb{Q}$ of x^4-2. Then determine the Galois group over $\mathbb{Q}$ of x^4-2, both as an abstract group and as a set of automorphisms. Give the lattice of subgroups

and the lattice of subfields. Make clear which subfield is the fixed field of which subgroup.
Note: This question came up in a slightly different form as Algebra Qualifying Exam Spring 2006 #2. This is an example we have discussed in lecture.

10. Algebra Qualifying Exam Fall 2008 #4
    Determine the splitting field over $\mathbb{Q}$ of $x^4 - 3$. Then determine the Galois group over $\mathbb{Q}$ of $x^4 - 3$, both as an abstract group and as a set of automorphisms. Give the lattice of subgroups and the lattice of subfields. Make clear which subfield is the fixed field of which subgroup.

11. Algebra Qualifying Exam Spring 2012 #5
    Let $L = \mathbb{Q}(\sqrt[6]{-3})$. Show that $L/\mathbb{Q}$ is Galois and $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$.

12. Algebra Qualifying Exam Fall 2014 #2
    Show that $\mathbb{Q}(\sqrt{2+\sqrt{2}})$ is a cyclic quartic extension of $\mathbb{Q}$, i.e., is a Galois extension of degree 4 with cyclic Galois group.
    Note: This is Exercise 14 in Section 14.2.

13. Algebra Qualifying Exam Fall 2014 #8
    Let p be prime. Prove that the Galois group for $x^p - 2$ over $\mathbb{Q}$ is isomorphic to the group of matrices
    $$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$
    with $a, b \in \mathbb{F}_p,\ a \neq 0$.
    Note: This is Exercise 5 in Section 14.2. A version of this problem arose as Algebra Qualifying Exam Winter 2000 #13. The specific instance of this problem with $p = 5$ came up as Algebra Qualifying Exam Spring 2009 #4 and as Algebra Qualifying Exam Spring 2001 #7.