Math 206C: Algebra
Practice Problems: Cyclotomic Fields
Please email `nckaplan@math.uci.edu` with questions.

In Lecture 25 we covered a lot of material about Cyclotomic Fields. In this document I will list some practice problems related to this topic. The first set of problems come directly from Lecture 25 and the next set of problems are additional exercises that you can do for additional practice.

1. Exercise 7 in Section 14.5
   This exercise discusses the *maximal real subfield* of $\mathbb{Q}(\zeta_n)$. In Lecture 25 we discussed two examples of these fields for $n = 5$ and $n = 7$.

2. Exercise 11 in Section 14.5
   This exercise was the primary focus of Lecture 25 Video 2.
   Note: One direction of this exercise (the difficult one) came up as Algebra Qualifying Exam Spring 2019 #6.

3. Exercise 13 in Section 14.5
   In this exercise you give the direct connection between $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ and the Chinese Remainder Theorem described in Corollary 27.

4. Exercise 33 in Section 14.6 and Exercise 11 in Section 14.7
   These exercises give different approaches to showing that $\mathbb{Q}(\zeta_p)$ contains either $\mathbb{Q}(\sqrt{p})$ or $\mathbb{Q}(\sqrt{-p})$ as a subfield. I would not recommend trying to solve these now, but you may want to think about them after the end of the course.

5. Algebra Qualifying Exam Spring 2011 #1
   Let p be an odd prime. Prove that $\mathbb{Q}(e^{2\pi i/p})$ contains a unique quadratic extension of $\mathbb{Q}$. For which p is this quadratic field contained in $\mathbb{R}$? Justify your answer.
   Note: The first part of this problem also came up as Algebra Qualifying Exam Fall 2007 #3a. This second part is closely related to the previous pair of exercises. You can solve the first part using the Galois Correspondence– I think the second part is tricky.

6. Algebra Qualifying Exam Fall 2016 #6
   Let $\zeta = e^{2\pi i/7} \in \mathbb{C}$ denote a primitive 7^{th} root of unity.

   (a) True/False: Every element in $\mathbb{Q}(\zeta)$ can be expressed **uniquely** in the form
   $$a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + a_4\zeta^4 + a_5\zeta^5 + a_6\zeta^6,$$
   where $a_0, \ldots, a_6 \in \mathbb{Q}$. Briefly explain.

   (b) Find the order of the element $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ induced by $\sigma\colon \zeta \to \zeta^2$. Briefly explain your answer.

   (c) Find the degree of the field extension $\mathbb{Q}(\zeta + \zeta^2 + \zeta^4)/\mathbb{Q}$. Explain your answer.

   Note: We solved this problem in Lecture 25 Video 3.

7. Algebra Qualifying Exam Spring 2013 #5b
   Find an irreducible cubic $f(x) \in \mathbb{Q}[x]$ whose roots generate the cubic subextension of $\mathbb{Q}(\zeta_7)/\mathbb{Q}$ where ζ_7 denotes a primitive 7^{th} root of unity in $\mathbb{C}$.
   Note: We solved this problem in Lecture 25 Video 3.

Additional Exercises

1. Algebra Qualifying Exam Fall 2013 #7
   Let E be the splitting field of $x^{21} - 1$ over $\mathbb{Q}$.

   (a) What is the degree $[E\colon \mathbb{Q}]$?

   (b) How many subfields does E have?

2. Algebra Qualifying Exam Fall 2004 #4
   Let F be the splitting field of $x^{10} - 1$ over $\mathbb{Q}$. Find $\mathrm{Gal}(F/\mathbb{Q})$, both as an abstract group, and as a group of explicitly described automorphisms of F.

3. Exercise 1 in Section 14.5
   This exercise builds on the example on page 598 about $\mathbb{Q}(\zeta_{13})/\mathbb{Q}$. You know generators of the subfields of $\mathbb{Q}(\sqrt{13})$ and in this exercise you compute their minimal polynomials.

4. Exercise 2 in Section 14.5
   This exercise emphasizes that the method of constructing an α_H for each subgroup $H \leq \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ really does require that you are working with the $\mathbf{p}^{\text{th}}$ cyclotomic field.

5. Exercise 3 in Section 14.5
   In Lecture 25, we found a quadratic equation satisfied by $\zeta_5 + \zeta_5^{-1}$. We know that $\mathbb{Q}(\zeta_5)$ is a quadratic extension of $\mathbb{Q}(\zeta_5 + \zeta_5^{-1})$. In this exercise you find a quadratic polynomial satisfied by ζ_5 with coefficients in $\mathbb{Q}(\zeta_5 + \zeta_5^{-1})$.

6. Exercise 10 in Section 14.5
   In Lecture 25 we stated (but did not carefully prove) that subfields of abelian extensions are abelian. (You should make sure that you understand why this is true.) In this exercise you see an interesting application of this fact.

7. Exercise 12 in Section 14.5.
   In this exercise we consider the Frobenius automorphism σ_p of $\mathbb{F}_q$ where $q = p^n$. We consider the characteristic polynomial of this linear transformation and determine when it is diagonalizable.