Math 206C: Algebra
Practice Problems Week 9

Please email nckaplan@math.uci.edu with questions.

We have had a lot of homework in this course, and I understand that many of you are busy with end-of-quarter things– exams, studying for Qualifying and Comprehensive Exams, etc.

This assignment is optional– you do not need to submit these problems.
I suggest that you try to solve some of them– they will be good practice for the Final Exam.

We will start by listing several problems that we discussed in Lectures 22 and 23.

1. Exercise 9 in Section 14.3
   Note: This is closely related to Algebra Qualifying Exam Spring 2006 #8, Algebra Qualifying Exam Fall 2004 #7, and Algebra Qualifying Exam Spring 2001 #5.
2. Algebra Comprehensive Exam Spring 2010 #F10
   Let p be a prime. What is the number of irreducible degree 4 polynomials in $\mathbb{F}_p[x]$?
3. Algebra Qualifying Exam Spring 2018 #9
   Let $\alpha \in \mathbb{F}_q^*$. Let K be a splitting field over $\mathbb{F}_q$ of $X^{q+1} - \alpha$. Prove that $[K : \mathbb{F}_q] = 2$.
4. Algebra Qualifying Exam Fall 2017 #6
   (a) Let $f(x) = x^{31} - 1 \in \mathbb{F}_2[x]$. What is $\mathrm{Gal}(f)$?
   (b) Let $f(x) = x^{31} - 1 \in \mathbb{F}_5[x]$. What is $\mathrm{Gal}(f)$?
5. Algebra Qualifying Exam Spring 2020 #7
   Consider the polynomial $g(x) = x^{20} + x^{10} + 1$.
   (a) Find the splitting field of $g(x)$ if we consider $g(x)$ as a polynomial in $\mathbb{F}_5[x]$.
   (b) Find the splitting field of $g(x)$ if we consider $g(x)$ as a polynomial in $\mathbb{F}_7[x]$.

Here are a few additional suggested problems on similar material.

1. Algebra Qualifying Exam Fall 2013 #6
   Let $f(x) = x^2 + x + 2 \in \mathbb{F}_5[x]$.
   (a) Prove that $f(x)$ is irreducible in $\mathbb{F}_5[x]$.
   (b) Explain why $f(x)$ divides the polynomial $x^{25} - x$ in $\mathbb{F}_5[x]$.
   (c) How many irreducible quadratic polynomials are there in $\mathbb{F}_5[x]$?
2. Algebra Qualifying Exam Spring 2019 #7
   Calculate the number of primitive elements of $\mathbb{F}_{27}$ over $\mathbb{F}_3$.
   Recall that if K/F is a field extension then $\alpha \in K$ is called a primitive element of K over F if and only if $K = F(\alpha)$.
3. Algebra Qualifying Exam Fall 1997 #7
   Let E be the splitting field of $x^{35} - 1$ over $\mathbb{F}_8$.
   Determine the cardinality $|E|$ of E. How many subfields does E have?

4. Exercise 3 in Section 14.3
   In this exercise you prove that an algebraically closed field is infinite, that is, every finite field is not algebraically closed.

5. Exercise 8 in Section 14.3
   This exercise asks about the family of polynomials $x^p - x - a \in \mathbb{F}_p[x]$, where $a \in \mathbb{F}_p^*$.
   These polynomials came up on HW4– see Exercise 5 in Section 13.5.

6. Exercise 5 in Section 14.3
   This exercise gives some practice writing down an explicit isomorphism between two finite fields of the same order.

7. Algebra Qualifying Exam Spring 2017 #10
   Let $K = \mathbb{F}_3(\sqrt{2})$ and let $f(x) = x^4 + 1 \in \mathbb{F}_3[x]$.
   (a) Show that K is the splitting field of f.
   (b) Find a generator α of the multiplicative group K^*.
   (c) Express the roots of f in terms of α.

Here are some additional practice problems that we can solve with the material we have developed recently.

1. Exercise 2 in Section 14.2
   Here we use ideas from Galois theory to compute the minimal polynomial of an element.

2. Exercise 2 in Section 14.4
   We know that every finite extension of $\mathbb{Q}$ is a simple extension. In this exercise you use Galois theory to find a primitive element.

3. Algebra Qualifying Exam Fall 2011 #4
   Determine the Galois closure F of the field $\mathbb{Q}(\sqrt{1+\sqrt{2}})$ over $\mathbb{Q}$. Determine all elements of the Galois group of the extension $F/\mathbb{Q}$ by describing their actions on the generators of F. Also describe this Galois group as an abstract group.
   Note: Part of this problem is Exercise 1 in Section 14.4.

4. Algebra Qualifying Exam Fall 2020 #7
   Let $E < F$ be a field extension of degree 5 and K the smallest subfield in the algebraic closure of E such that K is Galois over E and contains F. Show that the degree of K over E is at most 120.
   Note: I think this problem should have the additional assumption that F/E is a separable extension. (If it isn't, since Galois extensions are separable, no extension of E containing F will be Galois over E.)

5. Exercise 8 in Section 14.2
   Here we use the Fundamental Theorem of Galois theory to prove an interesting result about subfields of Galois extensions K/F where $[K\colon F]$ is a prime power.

6. Exercise 5 in Section 14.4
   Like the previous problem, you use ideas from Galois theory to prove something about field extensions of prime power degree.

7. Exercise 15 in Section 14.2
   In Section 13.2 we showed that every quadratic extension of a field of characteristic not equal to 2 is of a particular form– see page 522. In this exercise you use similar ideas to understand *biquadratic extensions*.

8. Algebra Qualifying Exam Winter 2003 #11
   Let K be the splitting field over $\mathbb{Q}$ of the polynomial

$$f(x) = (x^2 - 2x - 1)(x^4 - 1).$$

   Determine the Galois group G of $f(x)$ and determine all the intermediate fields explicitly.