Math 206C: Algebra
Midterm 1
Friday, April 23, 2021.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (3 Points)	
2 (3 Points)	
3 (6 Points)	
4 (8 Points)	
5 (8 Points)	
Total	

Problems	
6 (8 Points)	
7 (10 Points)	
8 (8 Points)	
9 (8 Points)	
10 (10 Points)	
Total	

Problems

1. True or False: Let A be any $n \times n$ matrix with entries in a field F. Then A is similar to its transpose, A^T.

2. True or False: Let F be any field and $p(x)$ be any monic polynomial of degree n in $F[x]$. There exists an $n \times n$ matrix A with entries in F that has minimal polynomial equal to $p(x)$.

3. (a) Let F be a field and K an extension of F. Define what it means for $\alpha \in K$ to be *algebraic* over F.
   (b) Define what it means for K/F to be algebraic.
   (c) Suppose $\alpha \in K$ is algebraic over F. Define the minimal polynomial of α over F, $m_{\alpha,F}(x)$.

4. Let A be an $n \times n$ matrix with entries in a field F.
   (a) Define the *trace* of A.
   (b) Define what it means for A to be *nilpotent*.
   (c) Prove that the trace of a nilpotent $n \times n$ matrix with entries in F is 0.

5. Suppose $A \in \mathrm{Mat}_3(\mathbb{C})$ has eigenvalues -1 and 2 (and no other eigenvalues). Let $c_A(x) \in \mathbb{C}[x]$ denote the characteristic polynomial of A, and $m_A(x) \in \mathbb{C}[x]$ denote the minimal polynomial.
   (a) Which pairs $(c_A(x), m_A(x))$ can occur?
   (b) For each pair that can occur, give an explicit example of a matrix A with those characteristic and minimal polynomials.

6. Find two matrices with entries in $\mathbb{C}$ that have the same characteristic polynomials and minimal polynomials but different Jordan canonical forms. Fully justify your answer.

7. Prove that for every $n \geq 2$ there exists an $n \times n$ nonsingular matrix $A \neq \pm I$ over $\mathbb{F}_3$ such that $I + A^2$ is its own inverse.

8. Prove that the characteristic of a finite field is prime.

9. Suppose K/F is a field extension of degree $[K\colon F] = p$ where p is prime. Show that for any $\alpha \in K$, either $F(\alpha) = F$ or $F(\alpha) = K$.

10. Let F be a field and let A and B be non-singular 3×3 matrices over F. Suppose that $B^{-1}AB = 2A$.
    (a) Find the characteristic of F.
    (b) If n is a positive or negative integer not divisible by 3, prove that the matrix A^n has trace 0.