# Math 206C: Algebra
## Midterm 1 Solutions

1. True or False: Let $A$ be any $n \times n$ matrix with entries in a field $F$.
   Then $A$ is similar to its transpose, $A^T$.

   **Solution**: This is true because $A$ and $A^T$ have the same invariant factors. We can determine the invariant factors of $A$ by diagonalizing the matrix $xI - A$ using elementary row and column operations. In order to diagonalize $xI - A^T$, we do the same sequence of operations, but every time we did a row operation to $A$ we do the corresponding column operation to $xI - A^T$, and every time we did a column operation to $xI - A$, we do the corresponding row operation.

2. True or False: Let $F$ be any field and $p(x)$ be any monic polynomial of degree $n$ in $F[x]$. There exists an $n \times n$ matrix $A$ with entries in $F$ that has minimal polynomial equal to $p(x)$.

   **Solution**: This is true. For example, the companion matrix of $p(x)$ has this property.

3. (a) Let $F$ be a field and $K$ an extension of $F$.
   Define what it means for $\alpha \in K$ to be *algebraic* over $F$.

   **Solution**: $\alpha$ is algebraic over $F$ if there exists a nonzero polynomial $f(x) \in F[x]$ with $f(\alpha) = 0$.

   (b) Define what it means for $K/F$ to be algebraic.

   **Solution**: $K/F$ is an algebraic extension if every element $\alpha \in K$ is algebraic over $F$.

   (c) Suppose $\alpha \in K$ is algebraic over $F$. Define the minimal polynomial of $\alpha$ over $F$, $m_{\alpha,F}(x)$.

   **Solution**: $m_{\alpha,F}(x)$ is the unique monic irreducible polynomial in $F[x]$ with $m_{\alpha,F}(\alpha) = 0$.

4. Let $A$ be an $n \times n$ matrix with entries in a field $F$.

   (a) Define the *trace* of $A$.

   **Solution**: The trace of $A$ is the sum of the diagonal entries of $A$.

   (b) Define what it means for $A$ to be *nilpotent*.

   **Solution**: $A$ is nilpotent if there is a positive integer $k$ such that $A^k = 0$

   (c) Prove that the trace of a nilpotent $n \times n$ matrix with entries in $F$ is 0.

   **Solution**: We showed in lecture that the trace of $A$ is the negative of the $x^{n-1}$ coefficient of the characteristic polynomial of $A$. Since $A$ is nilpotent, its minimal polynomial is a power of $x$. Since the characteristic polynomial divides some power of the minimal polynomial, $c_A(x) = x^n$. So the trace of $A$ is 0.

5. Suppose $A \in \mathrm{Mat}_3(\mathbb{C})$ has eigenvalues $-1$ and 2 (and no other eigenvalues). Let $c_A(x) \in \mathbb{C}[x]$ denote the characteristic polynomial of $A$, and $m_A(x) \in \mathbb{C}[x]$ denote the minimal polynomial.

(a) Which pairs $(c_A(x), m_A(x))$ can occur?

**Solution**: The characteristic polynomial $c_A(x)$ has degree 3. Its roots are the eigenvalues of $A$. So it has $-1$ and 2 as roots, and no others. Therefore, $(x+1)(x-2)$ divides $c_A(x)$, and since it has degree 3, there must be one other linear factor. Therefore, the characteristic polynomial is either $(x+1)^2(x-2)$ or $(x+1)(x-2)^2$.

The minimal polynomial divides the characteristic polynomial and the characteristic polynomial divides some power of the minimal polynomial. Therefore, every irreducible factor of the characteristic polynomial (in our case, $x+1$ and $x-2$) must divide the minimal polynomial. Therefore, we get the following list of possibilities for $(c_A(x), m_A(x))$:

i. $((x+1)^2(x-2), (x+1)^2(x-2))$,
ii. $((x+1)^2(x-2), (x+1)(x-2))$,
iii. $((x+1)(x-2)^2, (x+1)(x-2)^2)$,
iv. $((x+1)(x-2)^2, (x+1)(x-2))$.

(b) For each pair that can occur, give an explicit example of a matrix $A$ with those characteristic and minimal polynomials.

**Solution**: We first give the invariant factors for each of the possibilities described above and then give their Jordan canonical forms:

i. $(x+1)^2(x-2)$, $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

ii. $(x+1), (x+1)(x-2)$, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

iii. $(x+1)(x-2)^2$, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

iv. $(x-2), (x+1)(x-2)$, $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

6. Find two matrices with entries in $\mathbb{C}$ having the same characteristic polynomials and minimal polynomials but different Jordan canonical forms. Fully justify your answer.

**Solution**: The Jordan canonical form of a matrix is determined by its invariant factors. We need only give two matrices that have the same characteristic and minimal polynomial but different invariant factors. We consider two sets of possible invariant factors

$$(x-1), (x-1), (x-1)^2, \quad \text{and } (x-1)^2, (x-1)^2.$$

The minimal polynomial is the largest invariant factor, which is $(x-1)^2$ in both cases, and the characteristic polynomial is the product of all of the invariant factors, which is $(x-1)^4$ in both cases. A matrix with the first set of invariant factors is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and a matrix with the second set of invariant factors is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We note that both of these matrices are in Jordan canonical form.

7. Prove that for every $n \geq 2$ there exists an $n \times n$ nonsingular matrix $A \neq \pm I$ over $\mathbb{F}_3$ such that $I + A^2$ is its own inverse.

   **Note**: A few people found the wording of this problem confusing and instead interpreted it as asking for a matrix $A$ such that $A^{-1} = A^2 + I$. I apologize for the confusion. I will try to make thing kind of thing clearer in the future.

   **Solution**: $I + A^2$ is its own inverse if and only if $(I + A^2)^2 = A^4 + 2A^2 + I = I$, which means $A^4 + 2A^2 = 0$. This occurs if and only if the minimal polynomial of $A$ divides $x^2(x^2+2) = x^2(x+1)(x-1)$ in $\mathbb{F}_3[x]$.

   We note that $A$ is invertible if and only if 0 is not an eigenvalue of $A$, which occurs if and only if $x$ does not divide the characteristic polynomial of $A$. Since the minimal polynomial divides the characteristic polynomial of $A$, the only possibilities for the minimal polynomial of $A$ are $(x+1), (x-1)$ or $(x+1)(x-1)$. The only $A$ with minimal polynomial $x-1$ is $I$ and the only $A$ with minimal polynomial $x+1$ is $-I$.

   We need only show that for each $n \geq 2$ there is an $n \times n$ matrix with minimal polynomial $(x-1)(x+1)$. We can take the matrix with invariant factors $(x-1), \ldots, (x-1), (x-1)(x+1)$, where there are $n-2$ copies of $x-1$ at the beginning. An example of a matrix with these invariant factors is the one that is the $(n-2) \times (n-2)$ identity matrix in the upper left block, and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in the bottom $2 \times 2$ corner.

   **Note**: Some people solved this problem more directly by finding a matrix $A$ such that $A^2 + I = -I$. Over $\mathbb{F}_3$ this means that $A^2 = I$. So we need only find a matrix that squares to $I$ that is not $\pm I$. One choice is to take a diagonal matrix with 1 and $-1$ on the diagonal, at least one of each.

8. Prove that the characteristic of a finite field is prime.

   **Solution**: Let $F$ be a finite field. Since $F$ is finite, the elements $1, 1+1, 1+1+1, \ldots$ cannot be distinct. Let $n$ be the smallest positive integer such that $n \cdot 1$ is an element we have seen before. Then $n \cdot 1 = m \cdot 1$ for some positive integer $m < n$. This implies $n \cdot 1 - m \cdot 1 = (n-m) \cdot 1 = 0$. So the characteristic of $F$ is finite.

   Suppose the characteristic of $F$ is $n$. If $n = a \cdot b$ where $a, b > 1$, then

   $$n \cdot 1 = (a \cdot 1) \cdot (b \cdot 1) = 0,$$

   but since $a \cdot 1$ and $b \cdot 1$ are elements of a field, we must have $a \cdot 1 = 0$ or $b \cdot 1 = 0$, contradicting the assumption that $n$ was the **smallest** positive integer such that $n \cdot 1 = 0$.

9. Suppose $K/F$ is a field extension of degree $[K\colon F] = p$ where $p$ is prime. Show that for any $\alpha \in K$, either $F(\alpha) = F$ or $F(\alpha) = K$.

   **Solution**: We see that $F \subset F(\alpha) \subset K$ since $F(\alpha)$ is the smallest extension of $F$ containing $\alpha$ and $F$ and $K$ is **some** extension of $F$ containing $F$ and $\alpha$. Since degrees of field extensions are multiplicative, we have

   $$p = [K\colon F] = [K\colon F(\alpha)] \cdot [F(\alpha)\colon F].$$

   Since $p$ is prime and $[K\colon F(\alpha)], [F(\alpha)\colon F]$ are positive integers, the only possibilities are $[K\colon F(\alpha)] = p, [F(\alpha)\colon F] = 1$, in which case $F(\alpha) = F$, and $[K\colon F(\alpha)] = 1, [F(\alpha)\colon F] = p$, in which case $F(\alpha) = K$.

10. Let $F$ be a field and let $A$ and $B$ be non-singular $3 \times 3$ matrices over $F$. Suppose that $B^{-1}AB = 2A$.

    (a) Find the characteristic of $F$.

    **Solution**: $A$ is similar to $2A$, so in particular, they have the same determinant. We note that $\det(2A) = 2^3 \det(A)$, so we must have $2^3 = 1$ in $F$. This means that the characteristic of $F$ must be 7.

    (b) If $n$ is a positive or negative integer not divisible by 3, prove that the matrix $A^n$ has trace 0.

    **Solution**: Similar $n \times n$ matrices have the same characteristic polynomial, so they have the same trace (since the trace is the negative $x^{n-1}$ coefficient of the characteristic polynomial).

    Since
    $$(B^{-1}AB)(B^{-1}AB) = B^{-1}A^2B = (2A)(2A) = 4A^2,$$
    we see that $A^2$ is similar to $4A^2$.

    Similarly, $A^3$ is similar to $8A^3 = A^3$, but this tells us nothing.

In general, we see that $A^k$ is similar to $2^k A^k$. (This is true for positive or negative integers.)

This means that $\operatorname{tr}(A^k) = 2^k \operatorname{tr}(A^k)$, which is possible only when $2^k = 1$ in $F$ or when $\operatorname{tr}(A) = 0$. Since the characteristic of $F$ is 7, we see that $2^k = 1$ in $F$ if and only if $k$ is divisible by 3. Therefore, we see that when $k$ is not divisible by 3, $\operatorname{tr}(A^k) = 0$.

**Note**: This problem came from an old qualifying exam where it had a part (c): Prove that the characteristic polynomial of $A$ is $X^3 - a$ for some $a \in F$. I will give the solution here because it's pretty neat and may be useful for you to use this idea in the future.

**Solution**: Let

$$c_A(x) = x^3 + a_2 x^2 + a_1 x + a_0 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

We showed in the previous part that $-a_2 = \operatorname{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 0$. We now need only show that $a_1 = 0$.

We have $a_1 = (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$. Note that

$$0 = (\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2a_1.$$

We since the eigenvalues of $A^2$ are $\lambda_1^2, \lambda_2^2, \lambda_3^2$, we see that $\operatorname{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. We showed in the previous part that $\operatorname{tr}(A^2) = 0$ and we conclude that $2a_1 = 0$, so $a_1 = 0$.