Math 230A: Algebra

Final Exam

Thursday, December 8, 2022.

- You have **2 hours** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (8 Points)	
2 (8 Points)	
3 (8 Points)	
4 (18 Points)	
5 (10 Points)	
Total	

Problems	
6 (10 Points)	
7 (8 Points)	
8 (10 Points)	
9 (5 Points)	
10 (9 Points)	
Total	

Problems

1. Is the following statement True or False? Explain how you know that your answer is correct.
   Every element of the symmetric group S_6 has order at most 6.

2. Is the following statement True or False? Explain how you know that your answer is correct.

   For each $n \geq 3$, the automorphism group of the symmetric group S_n contains a subgroup isomorphic to S_n.

3. Is the following statement True or False? Explain how you know that your answer is correct.
   Any subfield of $\mathbb{R}$ must contain $\mathbb{Q}$.

4. Consider the subset R of $M_2(\mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

(a) Prove that R is a subring of $M_2(\mathbb{R})$.
(For this problem we are using the definition of a subring from Dummit and Foote where a subring does not necessarily have to contain the multiplicative identity of $M_2(\mathbb{R})$. Note that in this case R does contain the identity I_2, so don't worry about this issue.)

(b) Prove that R is a commutative ring with an identity, but is not an integral domain.

(c) Recall that an element x is an *idempotent* if $x^2 = x$. Find all idempotents in R.

(d) Define $\varphi\colon R \to \mathbb{R}$ by

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \to a - b.$$

Show that φ is a ring homomorphism.
(We use the Dummit and Foote definition of a ring homomorphism, so it is not required that φ takes the identity to the identity. In this case, the homomorphism does take the identity of R to the identity of $\mathbb{R}$, so you don't need to worry about this issue.)

(e) Determine $\ker(\varphi)$ as well as $R/\ker(\varphi)$.

(f) Is $\ker(\varphi)$ a prime ideal? Is $\ker(\varphi)$ a maximal ideal?
Explain how you know your answers are correct.

5. Let G be a group and M, N be normal subgroups of G.
   Prove that $G/(M \cap N)$ is isomorphic to a subgroup of $(G/M) \times (G/N)$.

6. For each of the following, either give an example (with an explanation) of such an ideal, or explain why such an example does not exist:

   (a) A prime ideal I in a finite commutative ring R with identity $1 \neq 0$ that is not a maximal ideal.

   (b) A prime ideal I in an integral domain R that is nonzero but not a maximal ideal.

7. Let G be a finite group and let p be a prime dividing $|G|$. Prove that there is a unique Sylow p-subgroup of G if and only if any Sylow p-subgroup of G is normal in G.

8. Let G be a group of order 12. Prove that G is isomorphic to a semidirect product $H \rtimes_\varphi K$ where $H, K \leq G$ are proper non-trivial subgroups of G.

9. Does there exist a group G and a subgroup $H \leq G$ such that G is a simple group and H is not a simple group? Either give an example or prove that no such example exists.

10. (a) State Cauchy's Theorem.

(b) State Cayley's Theorem.

(c) State the Third Isomorphism Theorem for Groups.