Math 230A: Algebra
Midterm 1
Wednesday, October 19, 2022.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (4 Points)	
2 (5 Points)	
3 (10 Points)	
4 (5 Points)	
5 (8 Points)	
Total	

Problems	
6 (10 Points)	
7 (8 Points)	
8 (10 Points)	
9 (8 Points)	
10 (10 Points)	
Total	

Problems

1. State the **Second Isomorphism Theorem**.

2. Let G be a group and H, K be subgroups of G. Consider the set $HK = \{hk \colon h \in H,\ k \in K\}$. Does HK always have to be a subgroup of G?
Either prove that the answer is yes, or give an example to show that it does not always have to be a subgroup.

3. Let G be a group and A be a nonempty subset of G.

   (a) Define the **centralizer** $C_G(A)$ of A in G.

   (b) Define the **normalizer** $N_G(A)$ of A in G.

   (c) Prove that $C_G(A)$ is a normal subgroup of $N_G(A)$.

   Note: You may use the fact that $C_G(A)$ and $N_G(A)$ are subgroups of G without proving it.

4. Are $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ isomorphic?
Either give an isomorphism between them or prove that no isomorphism exists.

5. Suppose G is a group acting on a set X. Prove that different orbits of this group action are disjoint and that these orbits partition the set X.

6. (a) Let G be a group and define the set of squares in G to be $S = \{g^2 \colon g \in G\}$.
Suppose $H \leq G$ is a subgroup of index 2. Prove that $S \subseteq H$.

(b) Define the set of cubes in G to be $C = \{g^3 \colon g \in G\}$.
Suppose $K \leq G$ is a subgroup of index 3. Do we have to have $C \subseteq K$?
Either prove this is always the case, or give an example to show that C is not always contained in K.

7. For each part of this problem, **explain how you know your answer is correct**.

   (a) For which positive integers n does S_n contain a subgroup isomorphic to $\mathbb{Z}/7\mathbb{Z}$?

   (b) For which positive integers n does S_n contain a subgroup isomorphic to $\mathbb{Z}/10\mathbb{Z}$?

8. Suppose G is a cyclic group. Prove that every subgroup H of G is cyclic.

9. (a) Suppose G is a group acting on a set X. (You may assume this is a left group action.) Define the stabilizer of x.

   (b) Let G be a group and $H \leq G$. We know that G acts on the set of left cosets of H in G by left multiplication. What is the stabilizer of the element $aH \in G/H$?
   Explain your answer.

10. Let G be a finite simple group having a subgroup H of prime index p. Show that p is the largest prime divisor of $|G|$.