Math 230a: Algebra

Final Exam

Friday, December 11 2015.

NAME:

- You have two hours for this exam. Pace yourself, and do not spend too much time on any one problem.
- Show your work and justify all of your answers. The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used. Do not use a calculator.
- If you need more room, use extra pages, and indicate clearly that you have done so.
- You may use results that we proved in lecture or on the homework without proving them here provided you clearly state the result your are using.

Problems	
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11	
Total	

1. For this problem you do not need to write any justification. You need only write yes or no for each part.

   Decide which of the following are subrings of $\mathbb{Q}$:

   (a) The set of all squares of rational numbers.

   (b) The set of all rational numbers with odd denominators (when written in lowest terms).

   (c) The set of all rational numbers with even denominators (which written in lowest terms).

   Decide which of the following are subrings of the ring of all functions form $[0, 1]$ to $\mathbb{R}$:

   (a) The set of all functions which have only a finite number of zeros, together with the zero function.

   (b) The set of all functions with an infinite number of zeros.

   (c) The set of all functions $f(x)$ such that $f(q) = 0$ for all $q \in \mathbb{Q} \cap [0, 1]$.

2. For this problem you do not need to write any justification. You need only write yes or no for each part.

   Decide which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:

   (a) $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to a$, (projection onto the 1,1 entry).

   (b) $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to a + d$, (the trace of the matrix).

   (c) $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to ad - bc$, (the determinant of the matrix).

   Decide which of the following are ideals of the ring $\mathbb{Z}[x]$:

   (a) The set of all polynomials whose constant term is a multiple of 3.

   (b) The set of all polynomials whose coefficients sum to zero.

   (c) The set of polynomials $p(x)$ such that $p'(x) = 0$ where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

3. Let I and J be ideals of R, a commutative ring with $1 \neq 0$.

(a) Prove that $I \cap J$ is an ideal.

(b) Show by example that the set of products $\{ab \mid a \in I,\ b \in J\}$ need not be an ideal.

(c) Show that the set of all finite sums of products of elements of the form ab with $a \in I$ and $b \in J$ is an ideal. This is called the product of I and J and is denoted IJ.

(d) Prove that $IJ \subseteq I \cap J$.

(e) Show by example that IJ and $I \cap J$ need not be equal.

4. Let H and K be groups, let $\varphi : K \to \mathrm{Aut}(H)$ be a group homomorphism, and as usual, identify H and K as subgroups of $G = H \rtimes_\varphi K$. Prove that $C_H(K) = N_H(K)$. (Recall that $C_H(K) = C_G(K) \cap H$.)

5. Classify up to isomorphism all groups of order 45.

6. If r, s are the usual generators for the dihedral group D_{2n} show that every subgroup of $\langle r \rangle$ is normal in D_{2n}.

7. Show that there are at least two non-isomorphic nonabelian groups of order 24, of order 30, and of order 40. (You should give two nonabelian groups of each order and prove that they are not isomorphic.)

8. Let G be a simple group containing an element of order 21. Prove that every proper subgroup of G has index at least 10.

9. Let $n \geq 2$ be an integer. Show that the only subgroup of index 2 of S_n is A_n.

10. Let F be a field (where $1 \neq 0$.) Show that $\langle F, + \rangle$ and $\langle F^{\times}, \cdot \rangle$ are not isomorphic as groups.

11. State a theorem that classifies all finite abelian groups up to isomorphism. This means that each finite abelian group should be isomorphic to exactly one group of your list. Use your classification to list abelian groups of order 24.