Math 230b: Algebra

In-Class Exam

Tuesday, February 23, 2016.

NAME:

- You have 80 minutes for this exam. Pace yourself, and do not spend too much time on any one problem (especially the true/false questions).
- Show your work and justify all of your answers. The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used. Do not use a calculator.
- If you need more room, use extra pages, and indicate clearly that you have done so.
- You may use results that we proved in lecture or on the homework without proving them here provided you clearly state the result your are using.

True/False	
1	
2	
3	
4	
5	
6	
Total	

Problems	
1	
2	
3	
4	
5	
6	
Total	

1 True/False: 3 Points Each

For this section you do not need to write any justification. You need only circle T or F for each question.

1. The field of fractions of the ring $2\mathbb{Z}$ is $\mathbb{Q}$.

   T **F**

2. Let R be a ring. Every proper ideal of R is contained in a unique maximal ideal of R.

   T **F**

3. Let $p \geq 3$ be prime and $R = \mathbb{Z}/p\mathbb{Z}$. Then $R[x]/(x^{p-1} + x^{p-2} + \cdots + 1)$ is a field.

   T **F**

4. Every nonzero prime ideal in a UFD is maximal.

T **F**

5. In a UFD every nonzero irreducible element is prime.

T **F**

6. If R is an integral domain and $f(x) \in R[x]$ is monic of degree d, then $f(x)$ has at most d distinct roots in R.

T **F**

2 Problems

Problem 1 is worth 15 points. The other five are worth 10 points each.

1. (a) Let R be a ring with a 1. Give the definition of a left R-module.

   (b) Let M, N be R-modules. Define what it means for a map $\varphi : M \to N$ to be an R-module homomorphism.

   (c) Give an explicit example of a map from one R-module to another that is a group homomorphism but not an R-module homomorphism.

   (d) Let R be a ring with a 1 and M an R-module. Prove that a nonempty subset $N \subset M$ is a (left) submodule of M if and only if $x + ry \in N$ for all $r \in R$ and for all $x, y \in N$.

2. (a) Prove that $\mathbb{Z}[x]$ is not a principal ideal domain. (If you prove this by giving an example of an ideal that is not principal, you must prove that it is not principal.)

   (b) Is it true that every ideal of $\mathbb{Z}[x]$ is generated by at most two elements? (If this is true, you should prove it. If it is false, give an example of an ideal that is not generated by two or fewer elements, but you do not need to prove that it has this property.)

3. Prove that a Euclidean domain is a principal ideal domain.

4. Prove that the multiplicative group of a finite field is cyclic.

5. (a) Let R be a UFD with field of fractions F. State Gauss' Lemma relating factorizations in $R[x]$ and $F[x]$.

   (b) Factor $2x^3 + 7x^2 - 2x - 1$ completely into irreducibles in $\mathbb{Q}[x]$.

6. Let R be a commutative ring with a 1 and M an ideal of R. Prove that M is maximal if and only if R/M is a field.