Math 230B: Algebra
Midterm #2
Wednesday, March 1, 2023.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (6 Points)	
2 (10 Points)	
3 (10 Points)	
4 (10 Points)	
5 (10 Points)	
6 (10 Points)	
Total	

Problems

1. (a) Let R be a commutative ring with $1 \neq 0$, let M be an R-module, and let $A \subset M$. Define what it means for A to span the R-module M.
(Equivalently, we could ask what it means for $RA = M$.)

(b) Let R be a commutative ring with $1 \neq 0$ and M, N be nontrivial R-modules. We know that $M \otimes_R N$ is spanned as an R-module by elementary tensors. Prove that every element of $M \otimes_R N$ is a **finite sum of elementary tensors**.

2. Let R be an integral domain and let M be an R-module. Recall that $\mathrm{Tor}(M)$ denotes all $m \in M$ such that there exists $r \in R \setminus \{0\}$ such that $r \cdot m = 0$. An R-module M is called *torsion-free* if $\mathrm{Tor}(M) = \{0\}$.

   (a) Prove that $\mathrm{Tor}(M)$ is an R-submodule of M.

   (b) Prove that $M/\mathrm{Tor}(M)$ is torsion-free.

3. Suppose V is a finite-dimensional vector space over a field F. Let $\mathrm{GL}(V)$ be the group of all invertible linear transformations from V to itself. Suppose G is a subgroup of $\mathrm{GL}(V)$, and define the ring

$$\begin{aligned} R \quad = \quad & \{\text{all linear transformations } T\colon V \to V \\ & \quad \text{such that } T(g(v)) = g(T(v)) \quad \text{for every } g \in G \text{ and } v \in V\}. \end{aligned}$$

Suppose further that if W is any subspace of V such that $g(W) \subseteq W$ for every $g \in G$, then either $W = 0$ or $W = V$.

Prove that if $T \in R$ and T is not the zero transformation, then T is invertible and $T^{-1} \in R$. **Hint**: If $T \in R$, what can you say about the kernel and image of T?

4. Suppose V is a vector space over a field F and that $T\colon V \to V$ is a linear transformation. Suppose that $v \in V$ and m is a positive integer such that $T^{m-1}(v) \neq 0$ and $T^m(v) = 0$.

   Prove that $v, T(v), T^2(v), \ldots, T^{m-1}(v)$ are linearly independent.

5. Let R be a commutative ring with a $1 \neq 0$ and M any (unital) R-module.
Prove that $R \otimes_R M \cong M$.

6. Let R be a commutative ring with $1 \neq 0$.
Prove that $R[x]$ is not a finitely generated R-module.