11 Infinite Sequences and Series

11.1 Sequences

Definition. A *sequence* is an ordered list of numbers. To denote the entire sequence we use either of the notations

$$(a_n)_{n=1}^{\infty} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

When it is unambiguous, we will simply write (a_n). In this course, all our sequences are *infinite*, and have an initial term. We refer to a_n as the *nth term* of the sequence.

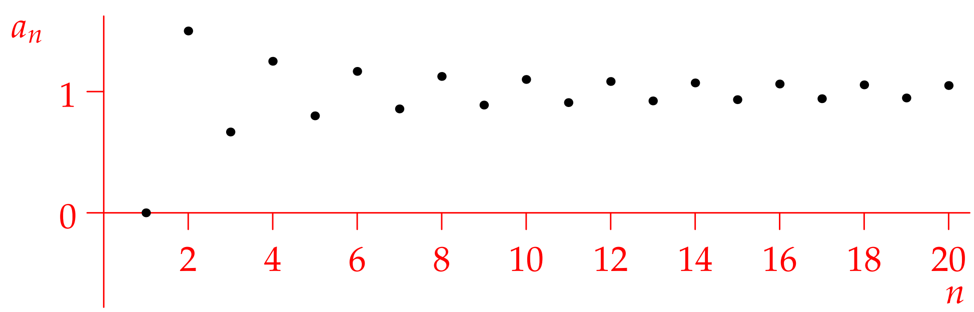
Examples

1. For example, the sequence $(1, 3, 5, 7, 9, 11, \dots)$ has nth term $a_n = 2n - 1$.

2. Another example is the sequence

$$(a_n)_{n=1}^{\infty} = (1, 3, 7, 15, 31, 63, 127, \dots) \quad \text{with } n\text{th term} \quad a_n = 2^n - 1$$

3. We can also plot sequences, just as if they are functions[1] For example, the sequence with nth term $a_n = 1 + \frac{(-1)^n}{n}$ is plotted below.



In the picture it seems clear that the values a_n get closer to 1, the larger n gets. This leads immediately to…

Definition. A sequence (a_n) has *limit* L, and we write

$$\lim_{n\to\infty} a_n = L$$

if we can force the values a_n to be as close as we like to L simply by choosing n to be large enough.[2] We say that (a_n) *converges* if there exists a limit, and *diverges* otherwise.

If should be clear that, as $n \to \infty$, we have that $\frac{1}{n}$ approaches zero. If $a_n = 1 + \frac{(-1)^n}{n}$ as in our example above, we immediately have

$$\lim_{n\to\infty} a_n = 1$$

[1] Indeed an alternative definition of a sequence is as a function whose domain is the natural numbers.

[2] A precise definition requires a strict idea of what 'close' means. See below.

More generally, if we can view a sequence in the form $a_n = f(n)$, where f is continuous, then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

All of the usual limit rules apply. In particular, if f happens to be differentiable, we may combine this with l'Hôpital's rule.

Examples

1. Let $a_n = ne^{-2n}$. Then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} xe^{-2x} = \lim_{x\to\infty} \frac{x}{e^{2x}}$$

is an indeterminate form of type $\frac{\infty}{\infty}$. Applying l'Hôpital's rule yields

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{1}{2e^{2x}} = 0$$

2. The sequence with nth term $a_n = 2^n - 1$ increases unboundedly as n increases. Therefore (a_n) diverges. In this case we would write

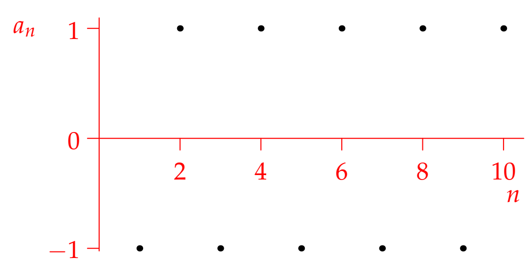
$$\lim_{n\to\infty} a_n = +\infty$$

and say that the sequence *diverges to infinity.*

3. The sequence $a_n = (-1)^n$ simply repeats the values ± 1: that is

$$(a_n)_{n=1}^{\infty} = (-1, 1, -1, 1, -1, 1, \dots)$$

These values are not getting closer to anything, so the sequence (a_n) diverges. In this case we would write



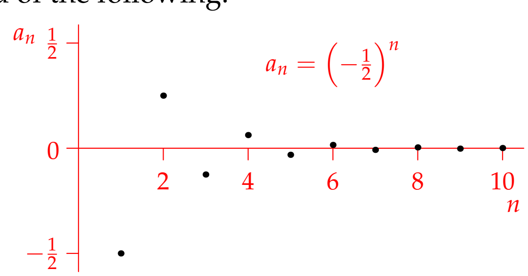
$$\lim_{n\to\infty} a_n = \text{DNE}$$

and say that the sequence *diverges by oscillation.*

4. A *geometric sequence* is a sequence whose successive terms have a constant ratio r. Any such sequence has nth term $a_n = ar^n$ where a and r are constants. For the sake of this example, we assume that $a = 0$, and consider how the limit of the sequence (r^n) depends on r.
It should be clear that if $-1 < r < 1$, then the sequence $(r, r^2, r^3, r^4, \dots)$ is getting closer to zero. Indeed, with a little thinking you should be convinced of the following:

$$\lim_{n\to\infty} r^n = \begin{cases} \text{DNE} & \text{if } r \le -1 \\ 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

In particular, (r^n) converges $\iff -1 < r \le 1$.



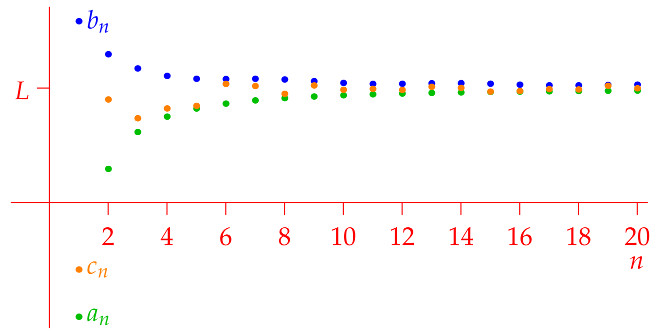
Comparing Sequences and showing the existence of limits

One of the purposes of sequences is to understand how process behave over long periods. A key idea is the ability to compare a complicated sequence to one which we understand completely, such as a geometric sequence (above). The first important result in this regard is the Squeeze Theorem, which is exactly analogous to the corresponding theorem for functions.

Theorem (Squeeze). *Suppose that* $(a_n), (b_n)$ *and* (c_n) *are sequences which satisfy the following properties.*

- *For all* n *we have* $a_n \le c_n \le b_n$.
- $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$.

Then (c_n) *also converges to* L.



Essentially we are saying that (c_n) is squeezed between the sequences (a_n), (b_n) which, since they are both approaching the same limit L, forces (c_n) to also approach L.

Example Since $-1 \le \sin n \le 1$, the sequence with nth term $c_n = \frac{\sin n}{n}$ is easily seen to satisfy

$$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

Since the left and right sides of this inequality both converge to zero, the squeeze theorem says that

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0$$

Most real-world problems cannot be solved exactly. For instance, when confronted with a sequence, one often needs to know that there is a limit *before* trying to find or approximate it. We therefore want to obtain properties of sequence which might tell us that a limit exists without computing it explicitly.

Definition. A sequence (a_n) is *non-decreasing* or *monotone-up* if

$$m > n \implies a_m \ge a_n$$

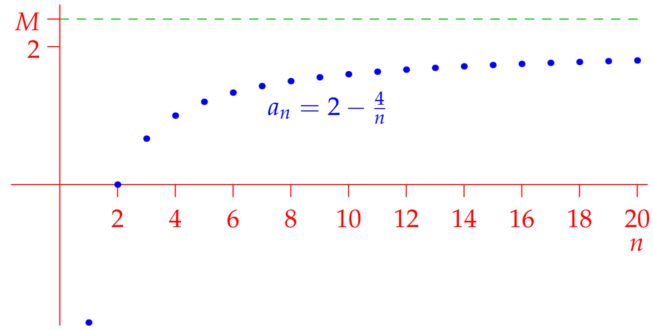
(a_n) is *non-increasing* or *monotone-down* if $m > n \implies a_m \le a_n$.
A sequence is *monotone* if it is either monotone down or up.
A sequence (a_n) is *bounded above* if all values of the sequence are less than or equal to some value. That is, if there exists some constant M for which

$$a_n \le M \text{ for all } n$$

A sequence is bounded below similarly. A *bounded* sequence is one which is bounded both above and below.

Theorem (Monotone Convergence). *Every bounded monotone sequence is convergent.*

The theorem is also true for monotone-up sequences which are bounded above, and for monotone-down sequences which are bounded below. Imagine a bumble-bee which never fliers lower than it currently is, but is trapped in a room below the ceiling. As time goes on, the bee's height must approach something which is less than or equal to the ceiling.



For example, the sequence $a_n = 2 - \frac{4}{n}$ is monotone-up and bounded above by $M = 2.4$ (or indeed by *any* $M \geq 2$). The theorem says that this sequence must converge. Of course we know that $\lim a_n = 2$ already, so the theorem is not useful for this example!

Advanced: The Monotone Convergence Theorem in Practice The use of the monotone convergence theorem is difficult in practice, since we are likely to be working with a sequence where the limit is not obvious. Consider the following sequence, defined inductively.

Let $a_1 = 2$ and define $a_{n+1} = \dfrac{a_n}{2} + \dfrac{1}{a_n}$ for each $n = 1, 2, 3, 4, \ldots$

The first few terms of the sequence are

$$(a_n) = (2, \tfrac{3}{2}, \tfrac{17}{12}, \tfrac{577}{408}, \tfrac{665857}{470832}, \ldots)$$

The sequence appears to be decreasing. If we can prove this, and that it is bounded below, then we will know that it converges. Unfortunately, proving these things requires a little creativity!

Bounded Below First notice that all terms of the sequence must be positive. Now observe that

$$a_{n+1}^2 = \left(\frac{a_n}{2} + \frac{1}{a_n}\right)^2 = \frac{a_n^2}{4} + 1 + \frac{1}{a_n^2} = \left(\frac{a_n}{2} - \frac{1}{a_n}\right)^2 + 2$$

It follows that $a_{n+1} > \sqrt{2}$ and so (a_n) is bounded below by $\sqrt{2}$.

Monotone-down Observe that

$$a_{n+1} - a_n = \frac{1}{a_n} - \frac{a_n}{2} = \frac{2 - a_n^2}{2a_n} < 0$$

since $a_n > \sqrt{2}$. Therefore (a_n) is a monotone-down sequence.

The theorem now says that the sequence has a limit $L = \lim\limits_{n\to\infty} a_n$. Armed with this knowledge we can actually compute L:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{a_n}{2} + \frac{1}{a_n} \implies L = \frac{L}{2} + \frac{1}{L} \implies L^2 = 2 \implies L = \pm\sqrt{2}$$

However, all the terms of the sequence are positive, so the limit cannot be negative. We conclude that $a_n \to \sqrt{2}$.

This sequence, and others like it have been used for thousands of years to obtain approximations to irrational numbers. Note that we really needed to show the existence of the limit *before* trying to calculate it. For example, naïvely substituting $L = \lim_{n\to\infty} b_n$ into the sequence defined by

$$b_2 = 2, \qquad b_{n+1} = \frac{3b_n}{2} - \frac{1}{b_n}$$

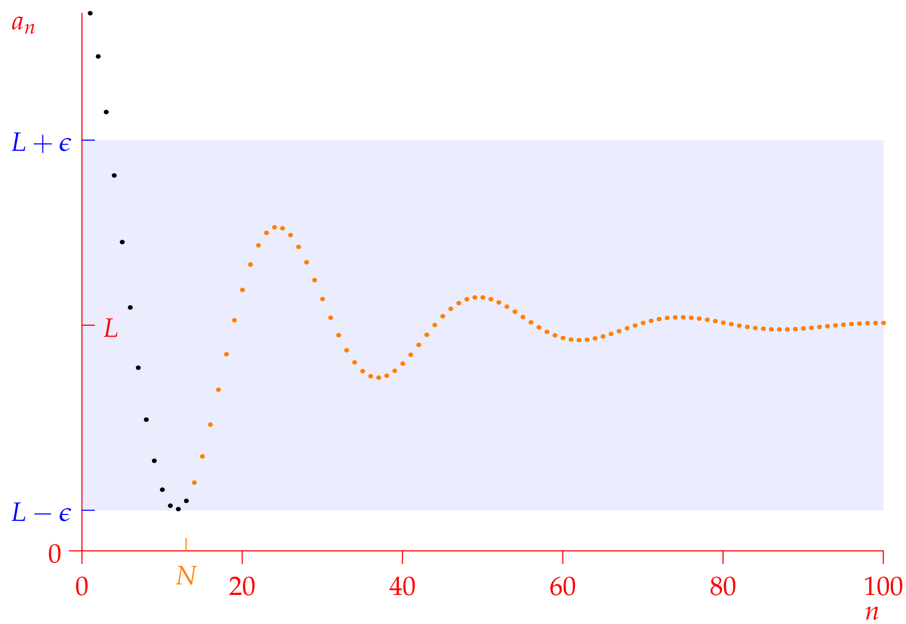
also yields the equation $L^2 = 2$. However, this sequence diverges to infinity!

Advanced: The Precise Definition of a Limit

Definition. We say that $\lim_{n\to\infty} a_n = L$ if, for each $\epsilon > 0$ there exists some N such that

$$n > N \implies |a_n - L| < \epsilon$$

This definition formalises the notion of 'close to' in the naïve definition of limit. The idea is that a_n is close to L if the difference between them is no more than some small value ϵ. The definition is saying that, regardless of how close (ϵ) we want the sequence to be to its limit, we are always able to find a tail of the sequence (all the terms after some a_N) closer to the limit than ϵ. The clickable picture below gives the idea: certainly, as we choose ϵ to be smaller, we are required to let N be larger so that the orange tail remains closer to L than ϵ.



Working directly with this definition is beyond the level of this course. Becoming comfortable with it is a critical part of upper-division mathematics.

Suggested problems

1. Determine whether each of the following sequences converges. It if does, find the limit.

(a) $a_n = n^{-2} + 2^{-n}$

(b) $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$

(c) $a_n = \cos(n^{-1})$

2. Find the limit of the following sequences. Show your working.

(a) $a_n = \frac{\sin(n^2)}{n^3}$.

(b) $a_n = n^2 e^{-2n}$.

3. A sequence is defined by the recurrence relation $a_{n+1} = \frac{1}{3}(a_n + 4)$ together with $a_0 = 1$.

(a) Suppose that $x < 2$. Show that $\frac{1}{3}(x + 4) < 2$.

(b) Use part (a) to show that (a_n) is bounded above.

(c) Show that (a_n) is increasing.

(d) By parts (a) and (b), and the monotone convergence theorem, (a_n) is convergent. What is it's limit?