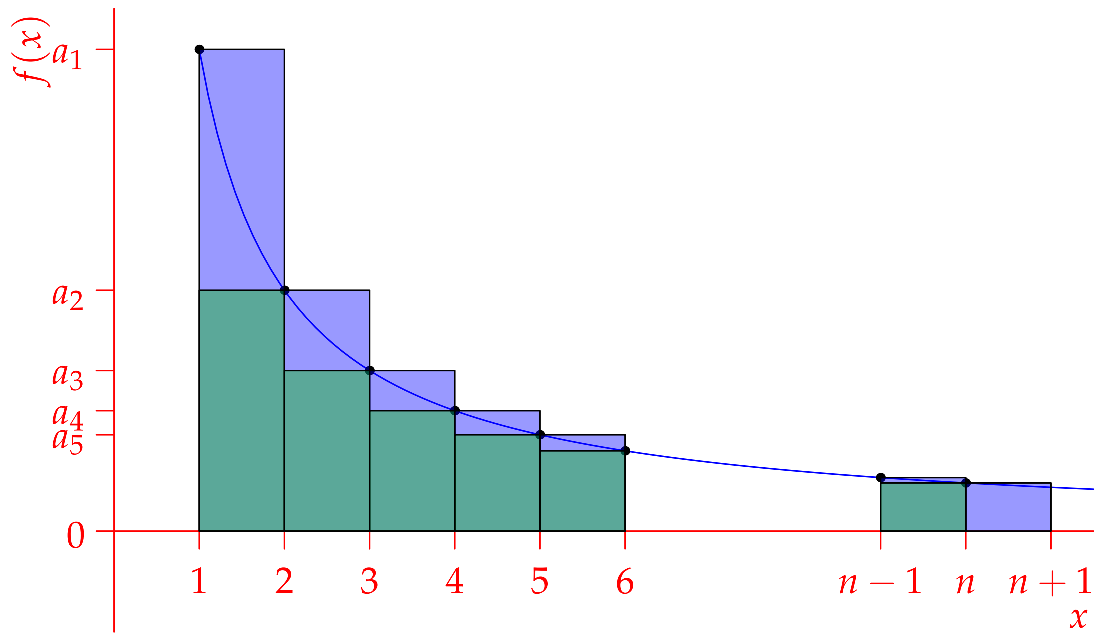
11.3 The Integral Test and Estimates of Sums

Much of the discussion of series involves methods or *tests* which may be applied to see if a series converges or diverges. Each test applies to different types of series and has different advantages and disadvantages. The integral test is our second of these (after the nth-term test). It formalizes the intuitive idea that integrals, being defined using limits of sums, should behave similarly to infinite series.

Consider the picture below. The graph of a decreasing, positive function f is drawn, where f has domain $[1, \infty)$. The sequence $(a_n)_{n=1}^{\infty}$ is defined by $a_n = f(n)$.



Notice that each of the rectangles has base 1 and height equal to one of the values a_i of the sequence. The areas of the rectangles are therefore equal to values of the sequence. In particular:

Green Rectangles The first has area a_2, and the last area a_n. Since all the rectangles lie below the curve $y = f(x)$ it is immediate that

$$\sum_{i=2}^{n} a_i \leq \int_1^n f(x)\,\mathrm{d}x \qquad (*)$$

Blue Rectangles The first has area a_1, and the last area a_n. Since the curbe $y = f(x)$ lies within the rectangles, we have

$$\int_1^{n+1} f(x)\,\mathrm{d}x \leq \sum_{i=1}^{n} a_i$$

Adding a_1 to both sides of $(*)$ and taking the limit as $n \to \infty$, we have proved the following:

Theorem (Integral Test)**.** *Suppose that f is a non-increasing, continuous, positive-valued function on the domain $[1, \infty)$. Then, for all $n = 1, 2, 3, 4, \ldots$, we have*

$$\int_1^{n+1} f(x)\,\mathrm{d}x \le \sum_{i=1}^{n} a_i \le a_1 + \int_1^n f(x)\,\mathrm{d}x$$

Moreover, $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^\infty f(x)\,\mathrm{d}x$ converges. In particular, if the infinite series converges, then

$$\int_1^\infty f(x)\,\mathrm{d}x \le \sum_{n=1}^{\infty} a_n \le a_1 + \int_1^\infty f(x)\,\mathrm{d}x$$

As with other tests, the initial term does not matter, we use $n = 1$ for brevity.

Examples

1. To test the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ we consider the function

   $$f(x) = \frac{1}{x^2+1}$$

   This is certainly continuous, and decreasing on the interval $[1, \infty)$. Moreover

   $$\int_1^\infty f(x)\,\mathrm{d}x = \tan^{-1} x \Big|_1^{\to\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

   It follows that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+1}$ converges and that its value satisfies

   $$\frac{\pi}{4} \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \frac{1}{2} + \frac{\pi}{4}$$

2. The function $f(x) = \frac{x}{x^2+1}$ has derivative $f'(x) = \frac{1-x^2}{(x^2+1)^2}$ which is negative for $x > 1$. Thus f is continuous and decreasing, whence we can apply the integral test. Since

   $$\int_1^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\ln(x^2+1)\Big|_1^{\to\infty} = +\infty$$

   we conclude that the infinite series diverges.

***p*-series**

The p-series are a family of infinite series. Together with the geometric series, they form the standard collection of series against which other, more complex, series may be compared.[1] For $p > 0$ constant, we consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

[1] I.e. using the comparison, ratio and root tests (later).

If $p = 1$ this is the *harmonic series*. Certainly the sequence defined by $a_n = \frac{1}{n^p}$ is decreasing. Recall our computation of the following indefinite integrals:

$$\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x = \begin{cases} \frac{1}{p-1} & \text{if } p > 1, \\ +\infty & \text{if } p \le 1 \end{cases}$$

Applying the integral test, we see that we have proved the following:

Theorem (Convergence of p-series)**.** *Let $p > 0$ be constant. The the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. Moreover, in such a case,*

$$\frac{1}{p-1} \le \sum_{n=1}^{\infty} \frac{1}{n^p} \le \frac{p}{p-1}$$

In particular, the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, while, if $p = 2$, we have

$$1 \le \sum_{n=1}^{\infty} \frac{1}{n^2} \le 2$$

Estimates of the growth rate of the harmonic series

Even though the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, we as ask how rapidly it does this. For example, how many terms of the series are required before the partial sum $s_n = \sum_{i=1}^{n} \frac{1}{i}$ exceeds 100?

According to the integral test,

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x \le s_n \le 1 + \int_1^n \frac{1}{x}\,\mathrm{d}x = 1 + \ln n$$

It follows that if s_n is to exceed 100, we certainly require

$$100 \le 1 + \ln n \iff n \ge e^{99} \approx 9.889 \times 10^{42}$$

Moreover, s_n is guaranteed to exceed 100 if

$$100 \le \ln(n+1) \iff n \le e^{100} - 1 \approx 2.688 \times 10^{43}$$

This is only an estimate, but the estimate is sickeningly large!

Suggested problems

1. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^2+9}$ converges.
2. Show that the series $\sum_{k=1}^{\infty} ke^{-k^2}$ satisfies the hypotheses of the integral test. Does it converge?
3. (Hard) For which values of p does the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^p}$ converge? Justify your answer using the integral test.