11.4 The Comparison Tests

The Comparison Test works, very simply, by comparing the series you wish to understand with one that you already understand. While it has the widest application of any of the series tests, and is the test on which all the remaining series tests are based, it is also the most difficult to use, given that it requires you to make an educated guess and sometimes requires significant creativity.

Theorem (Comparison test)**.** *Suppose that $\sum a_n$ and $\sum b_n$ are series of positive terms, such that $a_n \le b_n$ for all (sufficiently large) n. Then:*

- *If $\sum b_n$ converges, so does $\sum a_n$.*
- *If $\sum a_n$ diverges, so does $\sum b_n$.*

The proof relies on the monotone convergence theorem.

Proof. Suppose, for ease of notation that $1 \le n < \infty$, and that $0 < a_n \le b_n$. Since $\sum a_n$ and $\sum b_n$ are series of positive terms, they must either converge, or diverge to $+\infty$. If

$$s_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad t_n = \sum_{i=1}^{n} b_i$$

are the terms of the sequences of partial sums, then, being sums of *finitely many* positive terms, it is immediate that

$$0 < s_n \le t_n < \infty$$

and that the sequences (s_n) and (t_n) are both increasing.

- Suppose first that $\sum b_n = b$ converges. Then $\lim_{n\to\infty} t_n = b$. It follows that (s_n) is an increasing sequence, bounded above by b. Therefore (s_n) converges and so does $\sum a_n$.
- Instead suppose that $\sum a_n = \infty$ diverges. Then $\lim_{n\to\infty} s_n = +\infty$. It follows that $\lim_{n\to\infty} t_n = +\infty$, whence $\sum b_n$ diverges.

It is straightforward, though notationally messy to modify the proof to deal with $a_n \le b_n$ for 'sufficiently large' n.

$\blacksquare$

The most important thing about the comparison test is having a dictionary of well-understood series with which you can compare. The standards are:

Geometric Series $\sum r^n$ converges if and only if $-1 < r < 1$.

p-Series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.

Examples

1. It is clear that $n^2 + 3n + 2 \geq n^2$ for all $n \geq 1$, whence

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

   Since the right hand side is a convergent p-series, it follows that $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ converges.

2. Sometimes it is harder to see a suitable comparison. For example

$$\frac{\sqrt{n+1}}{2n^{3/2}-1} \geq \frac{\sqrt{n}}{2n^{3/2}-1} \geq \frac{\sqrt{n}}{2n^{3/2}} = \frac{1}{2n}$$

   Since $\sum \frac{1}{n}$ is a divergent p-series, it follows that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n^{3/2}-1} = +\infty$$

In general it can be very difficult to find a suitable series for comparison. A subtle change to the first example above makes this clear. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-3n+2}$$

where we start the sum at $n = 2$ to make sure that all terms are positive and we are not dividing by zero. We should believe that this series converges, since it still looks similar to $\sum \frac{1}{n}^2$, however

$$\frac{1}{n^2-3n+2} \quad \text{is not less than} \quad \frac{1}{n^2}$$

so we have to proceed differently. In this case we could say

$$\frac{1}{n^2-3n+2} = \frac{1}{(n-2)(n-1)} \leq \frac{1}{(n-1)^2}$$

whence

$$\sum_{n=2}^{\infty} \frac{1}{n^2-3n+2} \leq \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2}$$

which converges. Thankfully, this algebraic trickery can often be avoided by appealing to a more user-friendly result.

Theorem (Limit Comparison Test)**.** *Suppose that (a_n) and (b_n) are series of positive terms and consider the limit $c = \lim_{n\to\infty} \frac{a_n}{b_n}$, if it exists.*
If $c \neq 0$ and $c \neq \infty$, then either both $\sum a_n$ and $\sum b_n$ converge, or both diverge.

The basic idea for applying the limit comparison test to a series $\sum a_n$ is to imagine the terms a_n when n is very large and to choose b_n to be something simple which looks a bit like a_n. This is easiest to understand through examples.

Examples

1. The limit comparison test makes our previous example *much* easier. When n is very large, $n^2 - 3n + 2$ is not very different[1] to n^2. With reciprocals, the difference between $\frac{1}{n^2-3n+2}$ and $\frac{1}{n^2}$ is miniscule. If $a_n = \frac{1}{n^2-3n+2}$, we therefore compare to $b_n = \frac{1}{n^2}$. Since

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2 - 3n + 2} = 1$$

and $\sum \frac{1}{n^2}$ converges, it follows from the limit comparison test that $\sum a_n$ converges.

2. When n is large, $\frac{(n+2)3^n}{(2n+1)4^n}$ behaves a bit like $\left(\frac{3}{4}\right)^n$. Indeed

$$\lim_{n\to\infty} \frac{\frac{(n+2)3^n}{(2n+1)4^n}}{\left(\frac{3}{4}\right)^n} = \lim_{n\to\infty} \frac{n+2}{2n+1} = \frac{1}{2}$$

Since $\sum \left(\frac{3}{4}\right)^n$ is a convergent geometric series, by the limit comparison test we conclude that $\sum \frac{(n+2)3^n}{(2n+1)4^n}$ also converges.

3. If $a_n = (1 + \frac{1}{n})^2 e^{-n}$, then we compare with $b_n = e^{-n}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} (1 + \frac{1}{n})^2 = 1$$

Since $\sum e^{-n} = \sum (e^{-1})^n$ is a convergent geometric series, we conclude that $\sum a_n$ converges.

4. Suppose we want to decide on the convergence or divergence of $\sum a_n = \sum \sin \frac{1}{n}$. Think about what happens when n is large: clearly $\frac{1}{n}$ is small. Recalling that

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

we see that $\sin \frac{1}{n} \approx \frac{1}{n}$ for large n. This motivates us to compare with $\sum b_n = \sum \frac{1}{n}$. Indeed

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \qquad \text{(let } x = \tfrac{1}{n}\text{)}$$

Since $\sum \frac{1}{n}$ diverges, we conclude that $\sum \sin \frac{1}{n}$ also diverges.

5. Don't simply jump straight for the limit comparison test. If $\sum a_n = \sum \frac{3^n}{n \cdot 2^n}$, then there are two obvious divergent series we could compare with:, namely $\sum b_n = \sum \left(\frac{3}{2}\right)^n$ or $\sum d_n = \sum \frac{1}{n}$. Applying the limit comparison test with each of these yields

$$\lim \frac{a_n}{b_n} = 0 \qquad \text{and} \qquad \lim \frac{a_n}{d_n} = \infty$$

A modified version of the limit comparison test is given below, which allows for limits being 0 or ∞. However it is much easier to simply observe that

$$\frac{3^n}{n \cdot 2^n} \geq \frac{1}{n} \implies \sum \frac{3^n}{n \cdot 2^n} \geq \sum \frac{1}{n} = \infty$$

so that our original series diverges. This is just the (original) comparison test.

[1] If $n = 1000$, $n^2 - 3n + 2$ and n^2 differ by 2998. This seems large, but is small in comparison to n^2. Indeed, \$2,998 might seem like a lot of money, but how much would it matter to you if someone handed you \$1,000,000?

Advanced: The proof, and modifications of the Limit Comparison Test The proof of the limit comparison test intuitively comes from the following idea: if $0 < c < \infty$, then, for sufficiently large n, we have that $a_n \approx cb_n$, and so $\sum a_n \approx c \sum b_n$. To be precise, we have to use the ϵ-definition of limit.

Proof. If $0 < c < \infty$, then we may choose $\epsilon = \frac{c}{2}$ in the definition of limit. It follows that there exists a value N and bounds for which

$$\begin{aligned} n > N &\implies c - \epsilon < \frac{a_n}{b_n} < c + \epsilon \implies \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \\ &\implies \frac{c}{2} b_n < a_n < \frac{3c}{2} b_n \end{aligned}$$

Applying the comparison test, we see that, we have

$$\frac{c}{2} \sum_{n=N+1}^{\infty} b_n \le \sum_{n=N+1}^{\infty} a_n \le \frac{3c}{2} \sum_{n=N+1}^{\infty} b_n$$

Clearly both $\sum a_n$ and $\sum b_n$ converge, or both diverge. ∎

Since, for the limit comparison test, (a_n) and (b_n) are sequences of positive terms, it is possible that $c = 0$ or $c = \infty$.

- If $c = 0$ then, for large n the relation $a_n \le b_n$ holds, and the original conclusion of the comparison test applies.
  - If $\sum b_n$ converges, so does $\sum a_n$.
  - If $\sum a_n$ diverges, so does $\sum b_n$.
- If $c = \infty$ then, for large n the relation $a_n \ge b_n$ holds, and a reversed conclusion of the comparison test applies.
  - If $\sum a_n$ converges, so does $\sum b_n$.
  - If $\sum b_n$ diverges, so does $\sum a_n$.

It is unwise to attempt to memorize all of these possibilities. Instead, try to understand what the relationship $a_n \le b_n$ means for how the convergence/divergence of $\sum a_n$ and $\sum b_n$ must relate.

Suggested problems

1. Use a comparison test to decide whether the folowing series converge.

   (a) $\displaystyle\sum_{n=3}^{\infty} \frac{2^n}{n + 3^n}$.

   (b) $\displaystyle\sum_{j=2}^{\infty} (1 + j^{-2}) \cdot 4^{-j}$.

2. Does the series $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{1+n}{3+2n^2}}$ converge. Explain.

3. Suppose that $\sum a_n$ is a series of positive terms. Prove that $\sum a_n^2$ converges. (*Hint: Why is there a value N for which $n > N \Rightarrow 0 < a_n < 1$? Now apply a comparison…*)