11.5 Alternating Series

An *alternating sequence* is a sequence whose terms alternate between positive and negative. Often such sequences are written in the form

$$a_n = (-1)^n b_n \quad \text{or} \quad a_n = (-1)^{n+1} b_n$$

where (b_n) is a sequence of positive terms, although sometimes they are somewhat disguised. An *alternating series* is the sum of an alternating sequence. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots$$

is the *alternating harmonic series.*

The alternating series test is a convergence test which may be applied to alternating series. It is very easy to use.

Theorem (Alternating Series Test)**.** *Suppose that (b_n) is a decreasing sequence of positive values with limit zero. Then the alternating series $\sum (-1)^n b_n$ converges.*

Like the other series tests, it does not matter which value of n denotes the initial term. As long as a series is alternating and decreasing, then it will converge. Just make sure that you observe all these facts when using the alternating series test.

Examples

1. The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is certainly alternating, and the sequence $(\frac{1}{n})$ decreases with limit zero. The test applies and so the series converges.

2. Consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} e^{-(n^2+7n-2)}$$

Since the exponential term is always positive, this is certainly an alternating series. We should check that the exponential term is decreasing. For this, compute the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{-(x^2+7x-2)} = (-2x-7)e^{-(x^2+7x-2)} < 0 \quad \text{whenever} \quad x \geq 1$$

It follows that the alternating series test applies, and so the series converges.

3. Similarly, the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n (n^2+n)}{e^n}$$

is alternating and, since

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{(x^2+x)}{e^x} = \frac{(2x+1)e^x - (x^2+x)e^x}{e^{2x}} = \frac{2x+1-x^2+x}{e^x} = \frac{-x(x-3)-1}{e^x} < 0$$

if $x \geq 3$, the alternating series test applies.

4. Be careful! Not all alternating series converge!

$$\sum_{n=1}^{\infty}(-1)^{n-1}\sqrt{1+\frac{2}{n}}$$

is alternating, and $b_n = \sqrt{1+\frac{2}{n}}$ is decreasing. The series does not converge, since b_n does not converge to zero (nth term/divergence test).

Advanced: estimates of alternating series

If you read the proof of the alternating series test (below) you may be able to convince yourself of the following:

Theorem. *If $s = \sum(-1)^{n-1}b_n$ is a convergent alternating series, then the nth partial sum s_n is at most a distance b_{n+1} from the value s of the series. That is*

$$|s - s_n| \leq b_{n+1}$$

This result is mostly of academic interest, for alternating series typically converge to their limits very slowly...

Example It can be shown that the infinite series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{1+2n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

converges to $s = \frac{\pi}{4}$. How many terms would we need to sum in order to be sure that s_n is an approximation to s which is correct to 2 decimal places? To guarantee this, we solve

$$|s - s_n| \leq b_{n+1} < 0.01 \implies \frac{1}{1+2n} < \frac{1}{100} \implies n > 49.5$$

Advanced: proving the alternating series test

Like many similar proofs, this one relies on the monotone convergence theorem. We consider the sequence (s_n) of partial sums of a (decreasing) alternating series and show that half of this sequence (the even terms (s_{2m})) is decreasing and bounded below, while the other half (s_{2m+1}) is increasing and bounded above. Both halves converge. It remains to see that both halves converge to the same value. At all stages we need the fact that $a_n = (-1)^n b_n$ where b_n is a *decreasing* sequence.

Sketch Proof. For clarity, we assume that the series has the form $\sum_{n=0}^{\infty}(-1)^n b_n$ where (b_n) is a sequence which decreases to zero.
Consider the sequence of partial sums (s_n). Depending on whether n is even or odd, we have different expressions, whose terms may be grouped differently:

$$n = 2m \text{ even} \qquad s_{2m} = \sum_{i=0}^{2m}(-1)^i b_i = b_0 - (b_1 - b_2) - (b_3 - b_4) - \cdots - (b_{2m-1} - b_{2m})$$

$n = 2m + 1$ *odd* $\qquad s_{2m+1} = \sum_{i=0}^{2m+1} (-1)^i b_i = (b_0 - b_1) + (b_2 - b_3) + \cdots + (b_{2m} - b_{2m+1})$

Since (b_n) is decreasing, it follows that each of the bracketed terms above is *positive.* It follows that the subsequence (s_{2m}) is *decreasing* and that (s_{2m+1}) is *increasing*.
Moreover,

$$s_{2m} = (b_0 - b_1) + (b_2 - b_3) + \cdots + (b_{2m-2} - b_{2m-1}) + b_{2m} > 0$$
$$s_{2m+1} = b_0 - (b_1 - b_2) - (b_3 - b_4) - \cdots - (b_{2m} - b_{2m+1}) < b_0$$

(s_{2m}) is decreasing and bounded below, while (s_{2m+1}) is increasing and bounded above. The monotone convergence theorem says that both subsequences converge.
Finally,

$$s_{2m+1} - s_{2m} = -b_{2m+1} \to 0$$

so that both subsequences converge to the same limit.

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Suggested problems

1. (a) Show that $\sum_{k=2}^{\infty} \frac{(-1)^k}{k + \sqrt{k}}$ converges.

   (b) Why doesn't the alternating series test apply to the series $\sum a_j$, where

   $$(a_j) = \left(1, -\frac{3}{2}, \frac{1}{3}, -\frac{3}{4}, \frac{1}{5}, -\frac{3}{6}, \frac{1}{7}, -\frac{3}{8}, \frac{1}{9}, \ldots\right)?$$

2. Determine whether the following series converge.

   (a) $\sum_{k=3}^{\infty} \frac{(-1)^k (k-1)}{k^2 + 2}$

   (b) $\sum_{n=1}^{\infty} (-1)^{n+1} n^{1/n}$

3. You are given that $\pi^2 = \sum_{n=1}^{\infty} \frac{12(-1)^{n-1}}{n^2}$. How many terms of the series is it necessary to sum in order to approximate π^2 to within 0.03? Use a calculator to do so, and check your answer with the calculator's value for π^2.