11.6 Absolute Convergence and the Ratio and Root Tests

The most common way to test for convergence is to ignore any positive or negative signs in a series, and simply test the corresponding series of positive terms. Does it seem reasonable that the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \frac{6}{729} + \cdots$$

should say anything about whether the modified series (every third term is negative).

$$\frac{1}{3} + \frac{2}{9} - \frac{3}{27} + \frac{4}{81} + \frac{5}{243} - \frac{6}{729} + \cdots$$

converges? The fact that it does is somewhat remarkable. Before we understand this, we need to split the notion of convergence into two cases.

Definition. A series $\sum a_n$ is *absolutely convergent* if $\sum |a_n|$ converges.
A series $\sum a_n$ is *conditionally convergent* if it converges but not absolutely.

Examples

1. The series $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent, since the p-series $\sum \frac{1}{n^2}$ converges.

2. The alternating harmonic series $\sum \frac{(-1)^n}{n}$ is conditionally convergent: it converges, but the harmonic series $\sum \frac{1}{n}$ diverges.

Theorem. *If a series $\sum a_n$ is absolutely convergent then it is convergent.*

Proof. We use the fact that $a_n = (a_n + |a_n|) - |a_n|$. It follows, for any sequence (a_n), that

$$0 \le a_n + |a_n| \le 2|a_n|$$

Assuming that $\sum a_n$ is absolutely convergent (i.e. that $\sum |a_n|$ converges), we may apply the comparison test to see that

$$\sum (a_n + |a_n|)$$

converges. It follows, by the series laws, that

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is convergent. ∎

For a given series $\sum a_n$, we have shown that exactly one of three things must be true:

- $\sum a_n$ is absolutely convergent.
- $\sum a_n$ is conditionally convergent.
- $\sum a_n$ is divergent.

The theorem allows us to apply any of the tests we've seen that require only positive terms to *any* series. Such tests will only be able to show absolute convergence or divergence.

Example Show that $\sum a_n = \sum \frac{\sin(n^2)}{n^2}$ converges.
Since $0 \leq \left|\sin(n^2)\right| \leq 1$, we observe that

$$|a_n| \leq \frac{1}{n^2}$$

hence $\sum |a_n|$ converges by the comparison test. It follows that $\sum a_n$ converges absolutely: in particular, $\sum a_n$ converges.

The Ratio Test

The ratio test is perhaps the easiest of the convergence tests to use, but it is also one of the most likely to be inconclusive. It is particular useful for deciding on the convegence of series containing exponential and factorial terms.

Theorem (Ratio Test). *Let $\sum a_n$ be a series and let $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, if it exists. There are three possibilities:*

- *If $L < 1$ then $\sum a_n$ is absolutely convergent*
- *If $L > 1$ then $\sum a_n$ is divergent*
- *If $L = 1$ then the ratio test is inconclusive*

Sketch Proof. If $L < 1$ then $r = \frac{1+L}{2}$ lies half way between L and 1. Taking $\epsilon = \frac{1-L}{2}$ in the definition of limit, we see that, for n larger than some fixed N, we have

$$|a_{n+1}| \leq r\,|a_n| \implies |a_n| \leq |a_N|\, r^{n-N}$$

Now apply the comparison test to compare

$$\sum_{n=N+1}^{\infty} |a_n| \leq |a_N|\, r^{-N} \sum_{n=N+1}^{\infty} r^n$$

which is a convergent geometric series. It follows that $\sum a_n$ is absolutely convergent.
If $L > 1$, then for sufficiently large n we have $|a_{n+1}| \geq |a_n|$. It follows that the sequence (a_n) does not converge to zero, whence the nth term/divergence test says that $\sum a_n$ diverges.

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Examples

1. The ratio test is *useless* for series of rational expressions, as the limit will always be $L = 1$. Use comparison test instead. For example, we know that

$$\sum \frac{1}{n^2} \quad \text{converges, and} \quad \sum \frac{1}{n} \quad \text{diverges.}$$

The ratio test calculations in each case are

$$\lim_{n\to\infty} \left|\frac{1/n^2}{1/(n+1)^2}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2}\right| = 1 \quad \text{and} \quad \lim_{n\to\infty} \left|\frac{1/n}{1/(n+1)}\right| = 1$$

2. If $\sum a_n = \sum \frac{(-2)^n}{n!}$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}\cdot n!}{(n+1)!(-2)^n}\right| = \lim_{n\to\infty}\frac{2}{n+1} = 0$$

   It follows that $\sum a_n$ is absolutely convergent.

3. The ratio test is useful when you want to ignore polynomials. For example, if $\sum a_n = \sum \frac{3^n}{n^2 2^n}$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}\cdot n^2 2^n}{(n+1)^2 2^{n+1}\cdot 3^n}\right| = \lim_{n\to\infty}\frac{3n^2}{2(n+1)^2} = \frac{3}{2}$$

   Note the irrelevance of the polynomial terms. Since $\frac{3}{2} > 1$ we conclude that $\sum a_n$ diverges.

4. This final, tougher, example requires you to recall a limit from earlier in your calculus career.[1] If $\sum a_n = \sum \frac{n!}{n^n}$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\cdot n^n}{(n+1)^{n+1}\cdot n!}\right| = \lim_{n\to\infty}\left[\frac{n}{n+1}\right]^n = \lim_{n\to\infty}\frac{1}{\left[1+\frac{1}{n}\right]^n} = e^{-1}$$

   Sine $e^{-1} < 1$ we conclude that $\sum a_n$ is (absolutely[2]) convergent.

The Root Test

The root test is very similar to the ratio test. In the abstract it is slightly more useful, although it is typically less applicable to concrete series.

Theorem (Root Test). *Let* $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = |a_n|^{1/n}$, *if it exists. There are three possibilities, with the same conclusions as the ratio test:*

- *If* $L < 1$ *then* $\sum a_n$ *is absolutely convergent*
- *If* $L > 1$ *then* $\sum a_n$ *is divergent*
- *If* $L = 1$ *then the root test is inconclusive*

Sketch Proof. If $L < 1$ then $|a_n| \approx L^n$ for sufficiently large n. We now use the comparison test with the convergent geometric series $\sum L^n$.
If $L > 1$, then $\lim_{n\to\infty} |a_n| = L^n > 1$. In particular, the sequence (a_n) does not converge to zero, and so the series diverges. ∎

Because the root test involves taking nth roots, it is almost entirely useless! Unless a series is of the form $\sum(b_n)^n$, it is very unlikely that the root test will be at all useful.

[1] $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.
[2] Since each a_n is positive, absolute convergence is the same as convergence.

Examples

1. Consider $\sum a_n = \sum \frac{1}{n^n}$. Since

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\left(\frac{1}{n}\right)^n\right|^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and $0 < 1$, the root test quickly shows that $\sum \frac{1}{n^n}$ is (absolutely) convergent.
This example can also be easily done by comparison: if $n \geq 2$, then

$$\frac{1}{n^n} \leq \frac{1}{n^2} \implies \sum_{n=2}^{\infty} \frac{1}{n^n} \leq \sum_{n=2}^{\infty} \frac{1}{n^2} \quad \text{which converges.}$$

2. If $\sum a_n = \sum \left(\frac{1-n}{2n+1}\right)^n$, then we compute.

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1-n}{2n+1}\right| = \lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2}$$

The root test shows that $\sum a_n$ is absolutely convergent.

3. Finally, another example using the limit definition of e^x. If $\sum a_n = \sum \left(1 - \frac{1}{n}\right)^{n^2}$, then

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} < 1$$

whence $\sum a_n$ is (absolutely) convergent. $L = e^{-1}$ so convergent. Switch to + for counterexample.

To illustrate the proof of the root test for $L > 1$, consider modifying the last example. If $\sum b_n = \sum \left(1 + \frac{1}{n}\right)^{n^2}$, the root test produces a limit

$$\lim_{n\to\infty} |b_n|^{1/n} = e > 1$$

whence $\sum b_n$ diverges. However, it should be obvious that

$$b_n = \left(1 + \frac{1}{n}\right)^{n^2} > 1 \implies \lim_{n\to\infty} b_n \neq 0$$

so the series $\sum b_n$ diverges by the nth term/divergence test: the root test wasn't needed at all!

Suggested problems

1. (a) Explain the difference between absolute and conditional convergence.
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges. Is the convergence absolute or conditional? Explain your answer.

2. Use the Ratio or Root test to decide whether the following series converge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 3^n}{n!}$

(b) (Harder) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^{3n^2}$

3. (a) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$. Use the ratio test to decide whether this series converges.

   (b) Consider the following sequence:

$$a_n = \begin{cases} 2^{-n} & \text{if } n \text{ odd} \\ 3^{-n} & \text{if } n \text{ even} \end{cases}$$

   i. Attempt to apply the root test to $\displaystyle\sum_{n=1}^{\infty} a_n$. What happens?

   ii. Repeat (a) for the ratio test.

   iii. Does $\displaystyle\sum_{n=1}^{\infty} a_n$ converge? Prove it and, if it does, find its value.