5.2 Definite Integrals

While the last section discussed area, the algebraic constructionit contained is universal.

Definition. Suppose that f is a function defined on an interval $[a, b]$. Let n be a positive integer, define $\Delta x = \frac{b-a}{n}$, and let

$$x_i = a + i\Delta x = a + \frac{b-a}{n} i, \quad \text{for each} \quad i = 0, 1, \ldots, n$$

Choose *sample points* $x_i^* \in [x_{i-1}, x_i]$. A *Riemann Sum* is any expression of the form

$$\sum_{i=1}^{n} f(x_i^*) \Delta x$$

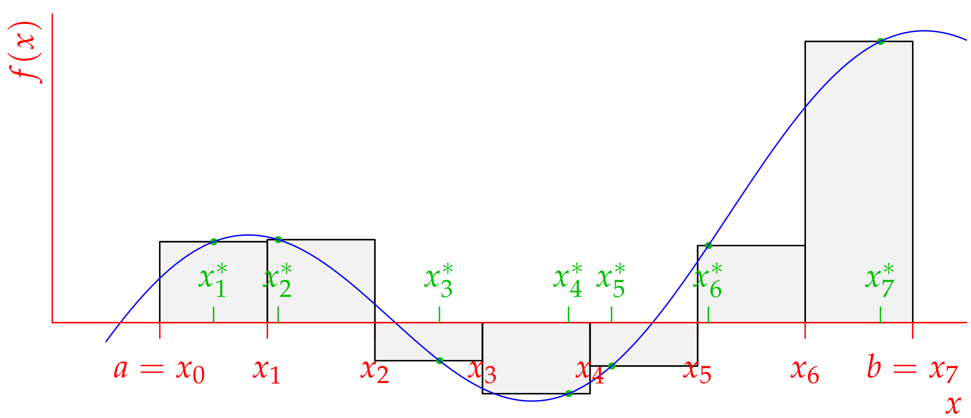
We say that the function f is *Riemann Integrable* on $[a, b]$ if

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

converges to the *same* value for *every* choice of sample points. In such a case the *definite integral of f from a to b* is[1]

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x$$

The integral sign $\int$ is a stylized S to remind you of the word "Σum."



The picture shows a choice of seven sample points. The sum of the (net) areas of the rectangles is a Riemann sum: if a rectangle is beneath the x-axis, then its area counts negatively.

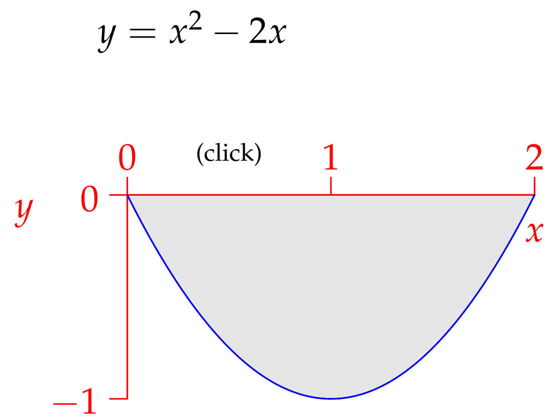
Theorem. *If f is continuous on $[a, b]$, or has only a finite number of jump discontinuities, then f is Riemann integrable on $[a, b]$.*

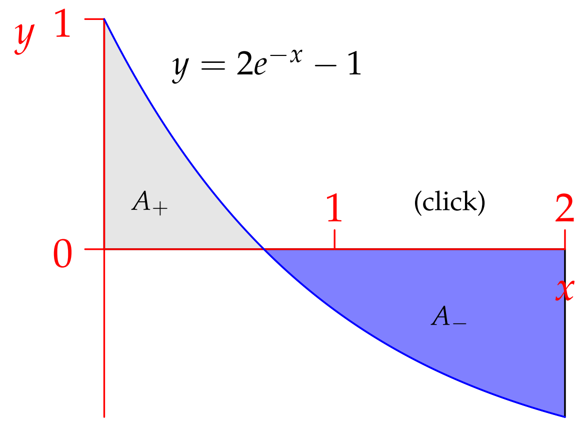
[1] Since the limit is the same for all sample points x_i^* we might as well take right endpoints $x_i^* = x_i$

Net area under a curve

If $f(x) \geq 0$, then $\int_a^b f(x)\,\mathrm{d}x$ = area under curve $y = f(x)$.
If $f(x) < 0$, then $\int_a^b f(x)\,\mathrm{d}x < 0$ is *negative* the area between the curve and the x-axis
In general $\int_a^b f(x)\,\mathrm{d}x$ = difference between the areas above and below the x-axis

$$\int_a^b f(x)\,\mathrm{d}x = A_+ - A_-$$





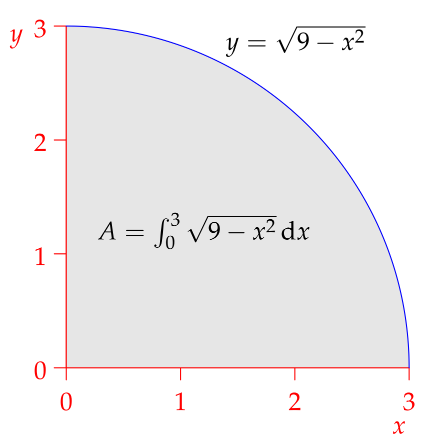
Examples

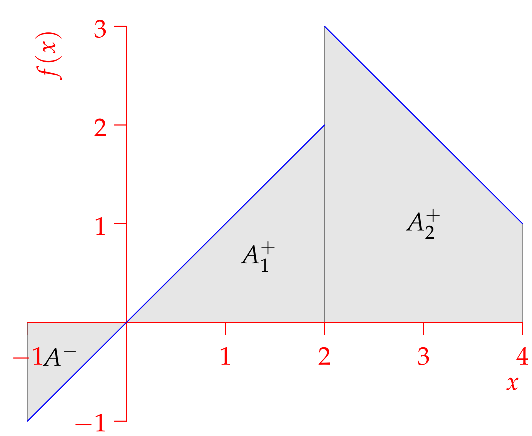
1. $\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$ represents the area of quarter circle of radius 3, hence

$$\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x = \frac{1}{4}\pi \cdot 3^2 = \frac{9}{4}\pi$$

2. If $f(x) = \begin{cases} x & x < 2 \\ 5-x & x \geq 2 \end{cases}$ then the integral $\int_{-1}^4 f(x)\,\mathrm{d}x$ can be computed by summing/subtacting the areas shown below:

$$\int_{-1}^4 f(x)\,\mathrm{d}x = A_1^+ + A_2^+ - A^- = 2 + 4 - \frac{1}{2} = \frac{11}{2}$$





The Midpoint Rule

For approximations, it is common to take sample points $x_i^* = \frac{1}{2}(x_{i-1} + x_i)$.

Examples

1. Using the midpoint rule to estimate $\int_0^1 x^2 \, \mathrm{d}x$ with 2 sample points yields

   $$\Delta x = \frac{1}{2}, \qquad x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1$$

   and sample points $x_1^* = \frac{1}{4}$ and $x_2^* = \frac{3}{4}$. It follows that

   $$\int_0^1 x^2 \, \mathrm{d}x \approx \left[\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 \right] \frac{1}{2} = \frac{5}{16}$$

   Given that the exact value for the integral is $\frac{1}{3}$, this is a very good approximation for very little work.

2. Use the midpoint rule with $n = 4$ to estimate $\int_3^5 x^{-3} \sin x \, \mathrm{d}x$.
   Here $\{x_1, x_2, x_3, x_4\} = \{3, 3.5, 4, 4.5, 5\}$ with $\Delta x = 0.5$, so

   $$x_1^* = 3.25, \quad x_2^* = 3.75, \quad x_3^* = 4.25, \quad x_4^* = 4.75$$

   Hence

   $$\begin{aligned} \int_3^5 x^{-3} \sin x \, \mathrm{d}x &\approx \sum_{i=1}^{4} f(x_i^*) \Delta x \\ &= \frac{1}{2} \left(\frac{\sin(3.25)}{3.25^3} + \frac{\sin(3.75)}{3.75^3} + \frac{\sin(4.25)}{4.25^3} + \frac{\sin(4.75)}{4.75^3} \right) \\ &= -0.01749 \text{ to 5d.p.} \end{aligned}$$

General properties of Integrals

Switching $b \leftrightarrow a$ changes the sign of $\Delta x = \frac{b-a}{n}$. Therefore

$$\int_b^a f(x) \, \mathrm{d}x = -\int_a^b f(x) \, \mathrm{d}x$$

In particular $\int_a^a f(x) \, \mathrm{d}x = 0$.

The following results are easy to confirm by drawing pictures. Thunk about heights of rectangles, and areas under curves.

Theorem. *If f, g are integrable on $[a, b]$ and c is constant, then*

1. $\int_a^b c \, \mathrm{d}x = c(b - a)$ — *area of rectangle height c, base $b - a$*

2. $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

3. $\int_a^b cf(x)\, dx = c \int_a^b f(x)\, dx$

4. $\int_a^c f(x)\, dx + \int_c^b f(x)\, dx = \int_a^b f(x)\, dx$

5. *If* $f(x) \geq 0$ *for all* $x \in [a,b]$ *then*

$$\int_a^b f(x)\, dx \geq 0$$

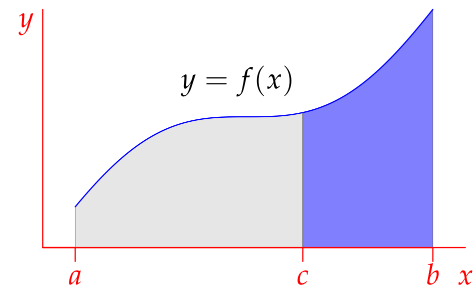
6. *If* $f(x) \geq g(x)$ *for all* $x \in [a,b]$ *then*

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$$

7. *If* $m \leq f(x) \leq M$ *on* $[a,b]$ *then*

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

4.



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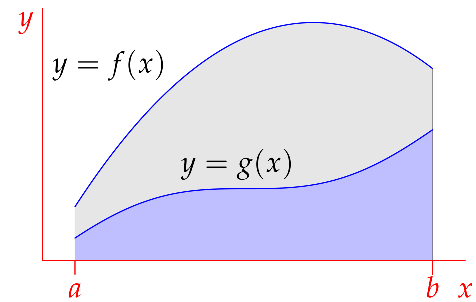

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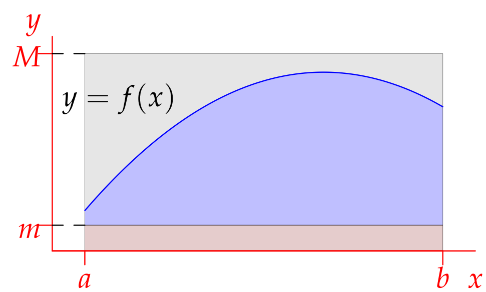

Suggested problems

1. Suppose you are given the following:

$$\int_0^3 f(x)\, dx = 4, \qquad \int_5^3 f(x)\, dx = -2, \qquad \int_0^5 g(x)\, dx = -5.$$

   Evaluate $\int_0^5 2f(x) - 3g(x)\, dx$.

2. A heater is switched on for 8 minutes. At time t minutes, the heater is using $E(t) = 200 - \frac{200}{1+t}$ kilojoules per minute. Use the midpoint rule with 4 subintervals to estimate the total energy consumed by the heater.

3. (Very hard!) Let f be the following function:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

   I.e. $f(3) = 3$, but $f(\sqrt{2}) = 0$.

(a) Consider the following Riemann sum for f on the interval $[0, 1]$.

$$\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n}.$$

Evaluate this sum (note that $\frac{i}{n}$ is a *rational number*).

(b) A property of the real numbers is that between any two rational numbers $p < q$ there exists an irrational number ξ. Show that there exists a Riemann sum for f on $[0, 1]$ with n subintervals for which all of the sample points x_i^* are irrational. What is the value of this Riemann sum?

(c) Comparing your answers to parts (a) and (b), *prove* that f is *not* Riemann integrable on the interval $[0, 1]$.