5.3 The Fundamental Theorem of Calculus

The Fundamental Theorem combines:

Anti-differentiation Find $F(x)$ such that $F'(x) = f(x)$

(Definite) Integration Find area under curve $y = f(x)$

It is termed *fundamental* because it provides the link between the two branches of calculus: differentiation and integration.

We first need to think of integrals as functions. We fix the lower limit of a definite integral to be a constant a and let the upper limit be variable. Thus if f is a function defined on an interval containing a and x, then

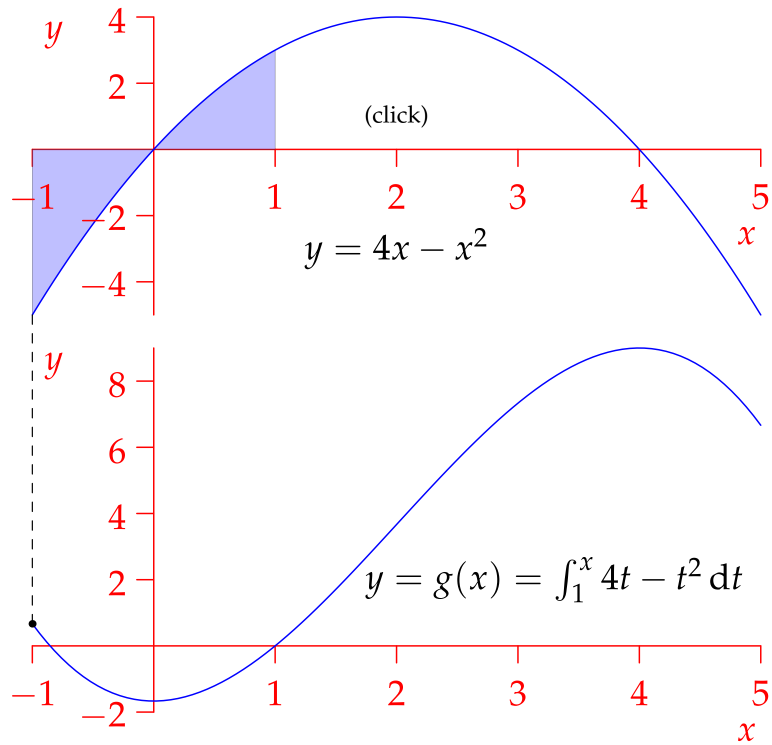
$$g(x) = \int_a^x f(t)\,\mathrm{d}t$$

is a function of x. The function g returns the *net* area under the curve $y = f(x)$ from a up to x. Recall the conventions from the previous section for how to understand values and net area: in particular, net area is *negative* if either

- $x < a$, or
- $f(t) < 0$

The animation should convince you that we can form such a function g: the lower graph returns the net area under the curve $y = 4x - x^2$ from 1 up to x. Some points of interest:

- $g(1) = 0$: no area!
- $g(4) > 0$
- $g(x)$ *decreases* when $x > 4$ since the net area for $x > 4$ is negative.
- $g(0) < 0$ since $0 < 1$: the area under the curve is being measured *backwards*. Indeed $g(0) = -\int_0^1 f(t)\,\mathrm{d}t$

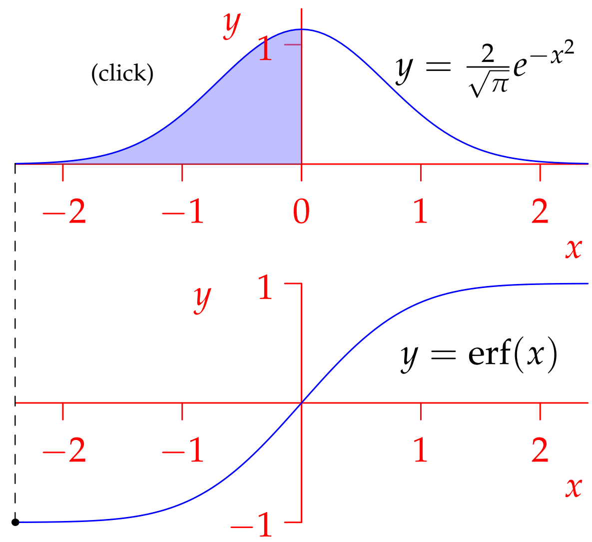


The Fundamental Theorem of Calculus can be seen in the picture; try sketching the derivative/slope of the lower graph, you should obtain the upper...

Example All manner of useful functions can be defined in this way. For example, the *error function*, of critical importance to probability, is defined by the integral

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t, \quad -\infty < x < \infty$$

The error function returns the net area under the curve $y = \frac{2}{\sqrt{\pi}}e^{-x^2}$ between 0 and x.
There is no way to express the error function in terms of the 'nice' functions you've thusfar met. It has to be defined using an integral (or something worse such as power series).



The first part of the Fundamental Theorem tells us how to differentiate such functions.

Theorem (FTC, part 1)**.** *Suppose that f is continuous on $[a,b]$. Then $F(x) := \int_a^x f(t)\,\mathrm{d}t$ is continuous on $[a,b]$, differentiable on (a,b), and its dertivative*[1] *is $F'(x) = f(x)$.*

We often write $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$. In essense, FTC part 1 says that if you integrate then differentiate, you return to what you started with.

Examples

1. The error function is differentiable for all real numbers x, and
$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\,\mathrm{d}t = \frac{2}{\sqrt{\pi}}e^{-x^2}$$

2. $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_3^x \cos(t^2)\,\mathrm{d}t = \cos(x^2)$

3. Switching limits: If the variable limit is at the bottom of the integral, we need to switch the limits, thus introducing a minus sign, before applying the Theorem. For example
$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^2 e^{\sin t}\,\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}x}\int_2^x e^{\sin t}\,\mathrm{d}t = -e^{\sin x}$$

4. Combining with the chain rule: If the variable limit is more complicated, then we need to use the chain rule. For example if we let
$$F(x) = \int_3^x (t^3-1)^5\,\mathrm{d}t$$
then
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_3^{e^{2x}} (t^3-1)^5\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}F(e^{2x}) = 2e^{2x}F'(e^{2x}) && \text{(chain rule)}\\ &= 2e^{2x}\left((e^{2x})^3-1\right)^5 = 2e^{2x}(e^{6x}-1)^5 && \text{(since } F'(x) = (x^3-1)^5\text{)}\end{aligned}$$

[1] Also on the open interval (a,b).

5. Separating. If both the lower and upper limits are variables, then we can separate into two integrals, switch the limits on one integral, and perhaps use the chain rule to finish things off. For example:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{x^3} \sin t\,\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{x^3} \sin t\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{0} \sin t\,\mathrm{d}t && \text{(split into two integrals)} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{x^3} \sin t\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{x^2} \sin t\,\mathrm{d}t && \text{(switch limits on second integral)} \\
&= 3x^2\sin(x^3) - 2x\sin(x^2) && \text{(apply FTC with chain rule)}
\end{aligned}$$

The choice of zero as the splitting point in the first line was irrelevant. We could have chosen any constant in the domain of $\sin t$.

The Theorem, Part 2 The second half of the fundamental theorem is more widely used. In essence, it says that if you differentiate and then integrate, you get back (almost) to where you started.

Theorem (FTC, part 2)**.** *Suppose that f is continuous on $[a, b]$ and that F is any antiderivative of f. Then*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

There are two alternative notations for evaluating the anti-deriative:

$$F(b) - F(a) = F(x)\Big|_a^b = [F(x)]_a^b$$

Examples

1. $\int_1^3 4x - x^2\,\mathrm{d}x = 2x^2 - \frac{1}{3}x^3\Big|_1^3 = \left(2\cdot 3^2 - \frac{3^3}{3}\right) - \left(2 - \frac{1}{3}\right) = \frac{22}{3}$

2. $\int_3^6 1 - x^{-2}\,\mathrm{d}x = x + x^{-1}\big|_3^6 = \left(6 + \frac{1}{6}\right) - \left(3 + \frac{1}{3}\right) = 2\frac{5}{6} = \frac{17}{6}$

3. A car has velocity $v(t) = 60(1 - e^{-240t})$ mph after t hours. How far does it travel in the first minute?
   $V(t) = 60t + \frac{60}{240}e^{-240t} = 60t + \frac{1}{4}e^{-240t}$ is an anti-derivative of $v(t)$, so the distance travelled is

$$\begin{aligned}
\int_0^{\frac{1}{60}} v(t)\,\mathrm{d}t &= \left[60t + \frac{1}{4}e^{-240t}\right]_0^{\frac{1}{60}} \\
&= \frac{1}{4}(3 + e^{-4}) \approx 0.755 \text{ miles}
\end{aligned}$$

Suggested Problems

1. Compute the following:

   (a) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_2^{x^3} \cos(3\sqrt{t})\,\mathrm{d}t$.

(b) $\dfrac{\mathrm{d}}{\mathrm{d}z}\displaystyle\int_{2z}^{3\sqrt{z}} e^{\sin u}\,\mathrm{d}u$.

(c) $\displaystyle\int_1^3 e^{2x} + x^2\,\mathrm{d}x$.

(d) $\displaystyle\int_{-\pi}^{2\pi} \cos^2 t\,\mathrm{d}t$.

2. Use the FTC part 1 to differentiate and hence find all the critical points of the function

$$y(x) = \int_0^x \frac{t(t+1)}{t^4+1}\,\mathrm{d}t$$

Now use the first derivative test from calc 1 to identify the type of each critical point.

3. In this question the function $u(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$ is the *unit step function.*

(a) Sketch the function $f(x) = xu(x)$ for $-1 \leq x \leq 1$.

(b) Show that

$$\int_{-1}^x tu(t)\,\mathrm{d}t = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

(c) Is $\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_{-1}^x tu(t)\,\mathrm{d}t = xu(x)$ for all x?

(d) Repeat parts (a-c) for $g(x) = e^x u(x)$. What is different?

Advanced/Non-examinable: Proofs, and why we need the continuity assumptions

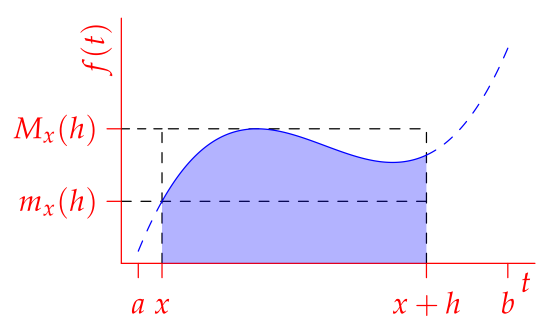
The statements of the two parts of the Fundamental Theorem both insist that f is continuous. To see why this is necessary we first consider the proofs. That of part 1 relies on the Extreme Value Theorem[2] which only applies to continuous functions. Part 2 is a corollary of part 1 and so also relies on the continuity assumption.

Proof of FTC, part 1. Let $x \in (a,b)$ and let $h > 0$ be small so that $x+h \in [a,b]$. Since f is continuous, the Extreme Value Theorem says that it is bounded on the interval $[x, x+h]$. It follows that there exist values

$$m_x(h) = \min\{f(t) : t \in [x, x+h]\}, \qquad M_x(h) = \max\{f(t) : t \in [x, x+h]\}$$

It follows that

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,\mathrm{d}t \quad \begin{cases} \leq hM_x(h) \\ \geq hm_x(h) \end{cases}$$



Since $h > 0$ we see that

[2] Covered in the discussion of continuity in differential calculus.

$$m_x(h) \le \frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \le M_x(h)$$

However f is continuous, and so therefore are the functions $m_x(h)$ and $M_x(h)$ (as functions of h). Clearly

$$\lim_{h\to 0^+} m_x(h) = f(x) = \lim_{h\to 0^+} M_x(h)$$

whence the Squeeze Theorem tells us that

$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t = f(x)$$

Repeating the calculation for $h < 0$ gives the Theorem.

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Proof of FTC, part 2. By FTC part 1, $F(x) := \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of $f(x)$. Clearly

$$F(b) - F(a) = \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x$$

The Mean Value Theorem says that any other anti-derivative of f on the interval $[a,b]$ differs from F by a constant: i.e. $G(x) = F(x) + c$. It follows that

$$G(b) - G(a) = (F(b)+c) - (F(a)+c) = F(b) - F(a)$$

is independent of the choice of anti-derivative.

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Just because the proof uses the continuity of f doesn't guarantee that we need it. To see that we do, we should investigate what happens when we try to apply the Fundamental Theorem to a discontinuous function. Try answering the questions below. The solutions are with those for the suggested problems.

1. Sketch the function $f(x) = \begin{cases} 1-x & x \le 1 \\ x & x > 1 \end{cases}$
2. Evaluate and sketch the graph of $F(x) := \int_0^x f(t)\,\mathrm{d}t$ for $0 \le x \le 2$.
3. Calculate $F'(x)$. Is it equal to $f(x)$? Explain.
4. Evaluate $\int_0^2 f(x)\,\mathrm{d}x$.
5. Show that $G(x) = \begin{cases} x - \frac{1}{2}x^2 & x < 1 \\ 1 + \frac{1}{2}x^2 & x > 1 \end{cases}$ is an anti-derivative of $f(x)$ whenever $x \neq 1$.
6. What is $G(2) - G(0)$? Explain.