5.4 Indefinite Integrals and the Net Change Theorem

The Fundamental Theorem of Calculus tells us that an anti-derivative of a continuous function f defined on an interval cointaining a may be written $F(x) = \int_a^x f(t)\mathrm{d}t$. Given this relationship between anti-derivatives and integrals we introduce a new notation for anti-derivatives:

$$F(x) = \int f(x)\mathrm{d}x \qquad \text{means the same thing as} \qquad F'(x) = f(x)$$

Definition. If f is a function which has an anti-derivative, then the *indefinite integral* of f is denoted $\int f(x)\mathrm{d}x$. This expression represents either:

1. All anti-derivatives of f.
2. A particular anti-derivative of f (rarely).

Be careful: the definite integral $\int_a^b f(x)\mathrm{d}x$ is a *number*, while the indefinite integral $\int f(x)\mathrm{d}x$ is a *function* or a family of functions: indeed

$$\int_a^b f(x)\mathrm{d}x = \left[\int f(x)\mathrm{d}x\right]_a^b$$

Table of Indefinite Integrals

We can rewrite the table of anti-derivatives from Section 4.9 as indefinite integrals:

$$\int kf(x)\mathrm{d}x = k\int f(x)\mathrm{d}x, \quad k \text{ constant}$$
$$\int f(x) + g(x)\mathrm{d}x = \int f(x)\mathrm{d}x + \int g(x)\mathrm{d}x$$
$$\int x^n\mathrm{d}x = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$$
$$\int \cos x\mathrm{d}x = \sin x \qquad \int \sin x\mathrm{d}x = -\cos x$$
$$\int \sec^2 x\mathrm{d}x = \tan x \qquad \int \csc^2 x\mathrm{d}x = -\cot x$$
$$\int \sec x \tan x\mathrm{d}x = \sec x \qquad \int \csc x \cot x\mathrm{d}x = -\csc x$$
$$\int x^{-1}\mathrm{d}x = \ln|x| \qquad \int e^x\mathrm{d}x = e^x$$
$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \sin^{-1} x \qquad \int \frac{1}{1+x^2}\mathrm{d}x = \tan^{-1} x$$

Examples

1. $\int 3x^2 - 1\mathrm{d}x = x^3 - x + c$
2. $\displaystyle\int 4x - 3e^x - \cos x\mathrm{d}x = 2x^2 - 3e^x - \sin x + c$
3. $\displaystyle\int 4x - 3\sin x - \cos x\mathrm{d}x = 2x^2 + 3\cos x - \sin x + c$

4. $\displaystyle\int \frac{2x^4 - 3\sqrt{x}}{x^3}\mathrm{d}x = \int 2x - 3x^{-5/2}\mathrm{d}x = x^2 - 3 \cdot \frac{-2}{3}x^{-3/2} + c = x^2 + 2x^{-3/2} + c$

5. To compute the net area under a curve, we have the option of first evaluating the indefinite integral. For example,

$$\int x^3 - 6\sin x \mathrm{d}x = \frac{1}{4}x^4 + 6\cos x + c$$

whence the net area under the curve $y = x^3 - 6\sin x$ between $x = 0$ and $x = \pi$ is

$$\frac{1}{4}x^4 + 6\cos x\Big|_0^\pi = \frac{1}{4}\pi^4 - 6 - (0+6) = \frac{1}{4}\pi^4 - 12$$

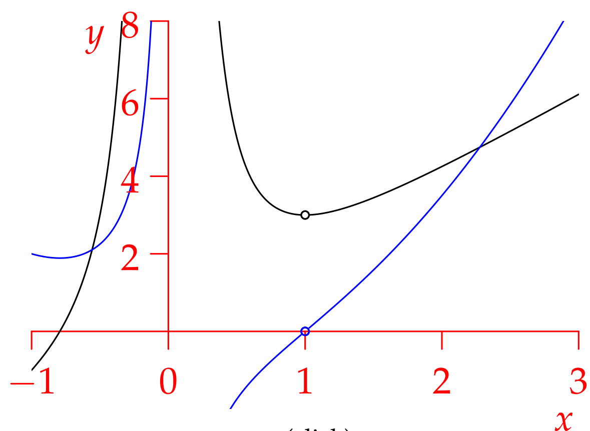
Discontinuous Functions

As we saw in the discussion of anti-derivatives, we must be careful of $\int f(x)\mathrm{d}x$ when f is discontinuous.

Example $f(x) = \dfrac{2x^2 - 2x}{x-1} + \dfrac{1}{x^2}$ is continuous except when $x = 0$ or 1.
If $x \neq 0, 1$ then $f(x) = 2x + \frac{1}{x^2}$, and so the general indefinite integral of f is

$$\int f(x)\mathrm{d}x = \int 2x + \frac{1}{x^2}\mathrm{d}x = x^2 - \frac{1}{x} + c$$
$$= \begin{cases} x^2 - \frac{1}{x} + c_1 & x < 0 \\ x^2 - \frac{1}{x} + c_2 & 0 < x < 1 \\ x^2 - \frac{1}{x} + c_3 & 1 < x \end{cases}$$

where the constants c_1, c_2, c_3 may be *different*.



(click)

The animation shows several anti-derivatives of f in blue against the original curve f in black

The Net Change Theorem

This is merely a rephrasing of the Fundamental Theorem, part 2. Recall that the derivative F' is the *rate of change* of F. The *net change* in the value of F over an interval $[a, b]$ is the difference $F(b) - F(a)$.

Theorem (Net Change Theorem)**.** *The integral of the rate of change of a function is the net change in that function:*

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)$$

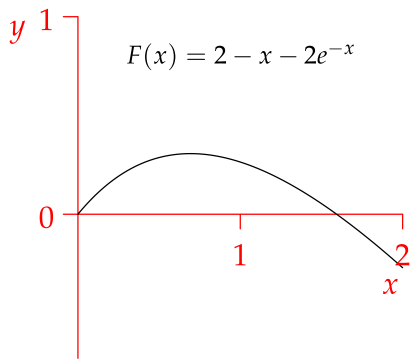
By contrast, the *total change*[1] is the integral $\int_a^b |F'(x)|\,\mathrm{d}x$

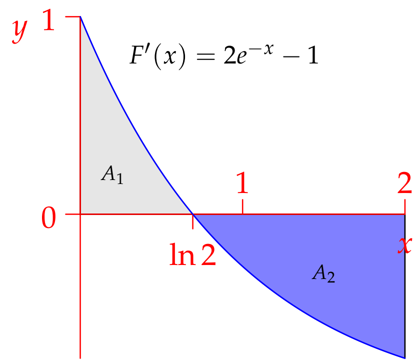
[1] Counts all changes in F positively

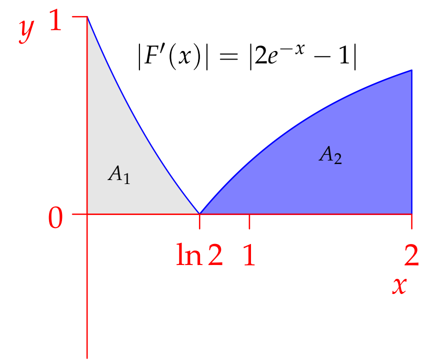
Example Let $F(x) = 2 - 2e^{-x} - x$, so that $F'(x) = 2e^{-x} - 1$.

The net change in $F(x)$ over $[0, 2]$ is clearly the difference $F(2) - F(0) = -2e^{-2}$. In terms of the pictures below and the net change theorem, we are computing the *difference* between the areas A_1 and A_2:

$$\int_0^2 F'(x)\mathrm{d}x = A_1 - A_2 = F(2) - F(0) = -2e^{-2}$$







To compute the *total change* of $F(x)$ over the interval, we need to find the *sum* of the areas A_1, A_2. This requires solving $F'(x) = 0$ to obtain $x = \ln 2$. The total change is then

$$\begin{aligned}\int_0^2 \left|F'(x)\right| \mathrm{d}x &= A_1 + A_2 = \int_0^{\ln 2} F'(x)\mathrm{d}x + \int_{\ln 2}^2 -F'(x)\mathrm{d}x \\ &= F(\ln 2) - F(0) - (F(2) - F(\ln 2)) = 2e^{-2} + 2 - 2\ln 2\end{aligned}$$

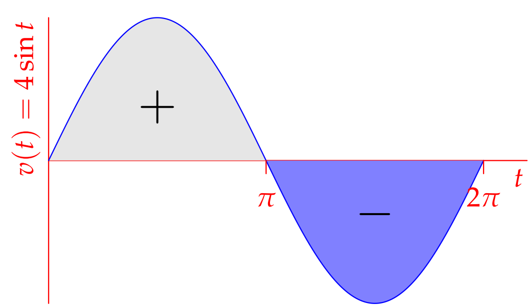
Distance and Displacement In Physics, the distinction between net and total change is exactly the discrepancy between displacement and distance, or between velocity and speed.[2]

Displacement Net change in position = integral of *velocity*.

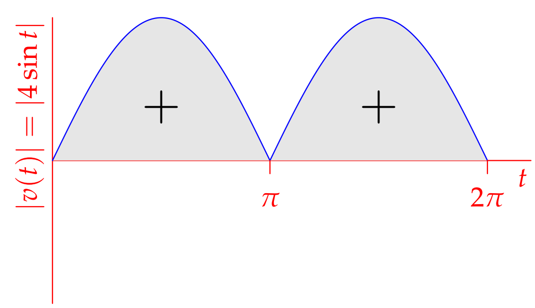
Distance travelled *Total change* in position = integral of *speed*.

If you walk directly the the store and back, your displacement would be zero, but your distance travelled would be positive.

For example suppose that a particle has velocity $v(t) = 4\sin t$ m/s at time t seconds.



Displacement = Net Change = 0



Distance = Total movement > 0

[2]Speed is the length of the velocity vector and is therefore ≥ 0. In one-dimension this is absolute value: speed $= |v(t)|$.

The displacement of the particle over the time interval $0 \le t \le 2\pi$ is then

$$\begin{aligned} s(2\pi) - s(0) &= \int_0^{2\pi} v(t)\mathrm{d}t = \int_0^{2\pi} 4\sin t\mathrm{d}t = -4\cos t\Big|_0^{2\pi} \\ &= -4 - (-4) = 0 \text{ m} \end{aligned}$$

The distance traveled over the same time period is

$$\begin{aligned} \int_0^{2\pi} |v(t)|\, \mathrm{d}t &= \int_0^{\pi} 4\sin t\mathrm{d}t + \int_{\pi}^{2\pi} -4\sin t\mathrm{d}t \\ &= -4\cos t\Big|_0^{\pi} + 4\cos t\Big|_{\pi}^{2\pi} = 4 - (-4) + 4 - (-4) = 16 \text{ m} \end{aligned}$$

Suggested problems

1. A particle starts at rest at $t = 0$. Its acceleration is given by

   $$a(t) = 2 - t \text{ m/s}^2.$$

   (a) Find the velocity at time t.
   (b) Find the displacement of the particle over the time interval $0 \le t \le 6$.
   (c) Find the distance traveled by the particle in the same time period.

2. A plane starts at 10,000 ft above sea level and its altitude changes at a rate $f(t)$ ft/min.

   (a) What is represented by the quantity $10,000 + \int_0^{12} f(t)\mathrm{d}t$?
   (b) If $\int_0^{20} f(t)\mathrm{d}t = -12,000$, what must have happened to the plane?