5.5 The Substitution Rule

Given the usefulness of the Fundamental Theorem, we want some helpful methods for finding anti-derivatives. At the moment, if an anti-derivative is not easily recognizable, then we are in difficulty. So how, for instance, do we find an integral such as $\int x^2 \sin(x^3)\,\mathrm{d}x$?

If it can be done, there are two possible approaches:

Guess and differentiate Make a guess, differentitate it to check, and modify your guess until you get it right!

Use a rule Recalling differential calculus, we might try to formulate some helpful rules based on the chain and product rules. The product rule will be resurrected later (as *integration by parts*). This section, on the substitution rule, explains how the chain rule may be applied to integral calculus.

Returning to our example, suppose that we are unable to make a sensible guess. The next approach is to try to subtitute away the ugliest expression. In this case, $\sin(x^3)$ is ugly so we define $u = x^3$. The expression $\sin u$ is certainly less daunting than $\sin(x^3)$. What's more, we know how to anti-differentiate $\sin u$. But how do we deal with the x^2 in the original integral? And what about the differential $\mathrm{d}x$?
These problems are dealt with simultaneously by differentiating our substitution

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 \implies x^2\,\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$$

We can now replace all the pieces in the original integral, integrate and substitute back at the end:

$$\int x^2 \sin(x^3)\,\mathrm{d}x = \int \frac{1}{3}\sin u\,\mathrm{d}u = -\frac{1}{3}\cos u + c = -\frac{1}{3}\cos(x^3) + c$$

This process is merely the chain rule written in a different way. Indeed we have the following Theorem:

Theorem (Substitution Rule)**.** *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then*

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

Proof. Let $u = g(x)$, and let F be an anti-derivative of f on the interval I, so that $F'(u) = f(u)$. The chain rule says that

$$\frac{\mathrm{d}}{\mathrm{d}x}F(g(x)) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x)$$

Any expression for a derivative may instantly be rephrased using indefinite integrals. Thus,

$$\int f(g(x))g'(x)\,\mathrm{d}x = F(g(x)) = F(u) = \int f(u)\,\mathrm{d}u$$

as required. ∎

You should explicitly write out your substitutions and their derivatives until you are comfortable with the method (and always when the calculation is long!). Obviously, there is no need to use substitution if you are capable of making a sensible guess. The first example that follows is in the master table of anti-derivatives so you should be able to state the answer without resorting to substitution.

1. To compute $\int \cos 3x \, \mathrm{d}x$ we let $u = 3x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3 \implies \mathrm{d}x = \frac{1}{3} \, \mathrm{d}u$. Therefore

$$\int \cos x \, \mathrm{d}x = \int \cos u \cdot \frac{1}{3} \, \mathrm{d}u = \frac{1}{3} \sin u + c = \frac{1}{3} \sin 3x + c$$

2. In the integral $\int \frac{x^3 \, \mathrm{d}x}{\sqrt{x^4-1}}$, the ugliest expression is in the square root. Therefore we let $u = x^4 - 1$, from which $\mathrm{d}u = 4x^3 \, \mathrm{d}x$. It follows that

$$\int \frac{x^3 \, \mathrm{d}x}{\sqrt{x^4 - 1}} = \int \frac{\mathrm{d}u}{4\sqrt{u}} = \int \frac{1}{4} u^{-1/2} \, \mathrm{d}u = \frac{1}{2} u^{1/2} + c = \frac{1}{2} \sqrt{x^4 - 1} + c$$

Making the wrong susbtitution It is very easy to make an unhelpful substitution. With some practice, you will quickly recognize when a substitution is going wrong, and try another! For example, suppose you had forgotten that $\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x = \sin^{-1} x + c$ and you wanted to compute the integral. Perhaps your first attempt is as follows:

- Try the substitution $u = 1 - x^2$.
- Then $x = \sqrt{1-u}$, and $\frac{\mathrm{d}u}{\mathrm{d}x} = -2x \implies \mathrm{d}x = -\frac{1}{2x} \, \mathrm{d}u = -\frac{1}{2\sqrt{1-u}} \, \mathrm{d}u$.
- Applying the substituion rule, we obtain

$$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x = \int -\frac{1}{2\sqrt{u}\sqrt{1-u}} \, \mathrm{d}u = -\frac{1}{2} \int \frac{1}{\sqrt{u - u^2}} \, \mathrm{d}u$$

- This looks even worse than the intergal we started with!

Instead, a different substitution prevails:

- We hope to remove the square root in the integrand by using the identity $1 - \sin^2 \theta = \cos^2 \theta$. We therefore try the substitution[1] $x = \sin \theta$.
- Then $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \cos \theta \implies \mathrm{d}x = -\cos \theta \, \mathrm{d}\theta$.
- Applying the substituion rule, we obtain

$$\int \frac{1}{\sqrt{1 - x^2}} \, \mathrm{d}x = \int \frac{1}{\sqrt{1 - \sin^2 \theta}} \cos \theta \, \mathrm{d}\theta = \int \frac{\cos \theta}{\cos \theta} \, \mathrm{d}\theta = \int \mathrm{d}\theta = \theta + c = \sin^{-1} x + c$$

as expected. Note that the square-root gives $+\cos \theta$. This is since the range of $\theta = \sin^{-1} x$ is the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, on which $\cos \theta > 0$.

[1]Substitutions can use *any letter*, not just u!

Substitutions in Definite Integrals

The substitution rule can be applied directly to definite integrals. The important point is that you must *change the limits!*

Theorem. *If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then*

$$\int_a^b f\big(g(x)\big)g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

Example To evaluate $\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$ we substitute $u = 2x+1$. Then

$$\mathrm{d}u = 3\,\mathrm{d}x, \qquad u(0) = 1, \quad u(4) = 9$$

It follows that

$$\begin{aligned} \int_0^4 \sqrt{2x+1}\,\mathrm{d}x &= \int_1^9 \frac{1}{3}\sqrt{u}\,\mathrm{d}u && \text{(substitute \textit{and} change limits)} \\ &= \frac{1}{3}\cdot\frac{3}{2}u^{3/2}\Big|_1^9 && \text{(find anti-derivative)} \\ &= \frac{1}{2}\left(9^{3/2} - 1^{3/2}\right) = 13 && \text{(evaluate and simplify)} \end{aligned}$$

Notice that once we substitute and change the limits, we never see x again.

An alternative to changing the limits is to first compute the *indefinite integral* then substitute back. For example:

$$\int \sqrt{2x+1}\,\mathrm{d}x = \int \frac{1}{3}\sqrt{u}\,\mathrm{d}u = \frac{1}{3}\cdot\frac{3}{2}u^{3/2} + c = \frac{1}{2}(2x+1)^{3/2} + c$$

Therefore

$$\int_0^4 \sqrt{2x+1}\,\mathrm{d}x = \frac{1}{2}(2x+1)^{3/2}\Big|_0^4 = \frac{1}{2}\left(9^{3/2} - 1^{3/2}\right) = 13$$

Both of these methods are acceptable. What is incorrect is to mix them. In what follows, the errors are in red.

$$\begin{aligned} \int_0^4 \sqrt{2x+1}\,\mathrm{d}x &= \int_{\color{red}0}^{\color{red}9} \frac{1}{3}\sqrt{u}\,\mathrm{d}u && \text{(substitute \textit{without changing limits})} \\ &= \frac{1}{3}\cdot\frac{3}{2}u^{3/2}\Big|_{\color{red}0}^{\color{red}9} && \text{(find anti-derivative)} \\ &= \frac{1}{2}(2x+1)^{3/2}\Big|_0^9 && \text{(substitute back)} \\ &= \frac{1}{2}\left(9^{3/2} - 1^{3/2}\right) = 13 && \text{(evaluate and simplify)} \end{aligned}$$

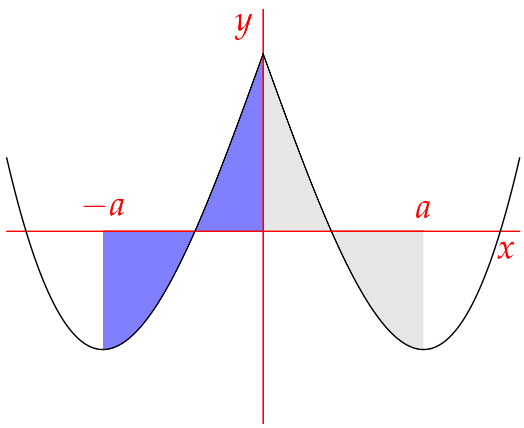
Since you obtained the correct solution (13), you'd likely assume you did the calculation correctly. Don't make this mistake!

Odd and Even Functions

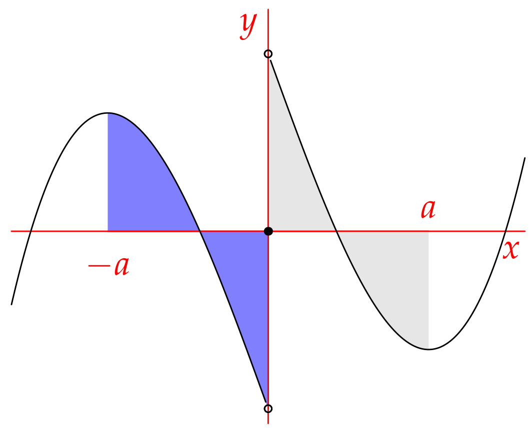
When functions have symmetry, we can often use the symmetry to compute integrals very quickly. Functions which are odd or even are particular straightforward, if integrated over a *symmetric interval* $[-a, a]$.

Theorem. *Suppose that f is continuous on $[-a, a]$.*

1. *If f is even ($f(-x) = f(x)$) then* $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x$
2. *If f is odd ($f(-x) = -f(x)$) then* $\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$



An Even Function



An Odd Function

Proof. Let f be odd and substitute $u = -x$. Then $\mathrm{d}u = -\,\mathrm{d}x$, $u(a) = -a$ and $u(-a) = a$. Therefore

$$\begin{aligned}
\int_{-a}^{a} f(x)\,\mathrm{d}x &= \int_{a}^{-a} f(-u)(-\,\mathrm{d}u) = -\int_{-a}^{a} f(-u)(-\,\mathrm{d}u) && \text{(swap limits)}\\
&= \int_{-a}^{a} f(-u)\,\mathrm{d}u && \text{(cancel negative signs)}\\
&= \int_{-a}^{a} -f(u)\,\mathrm{d}u && \text{(since } f \text{ is odd)}\\
&= -\int_{-a}^{a} f(x)\,\mathrm{d}x && \text{(since } u \text{ is a dummy variable)}
\end{aligned}$$

Therefore $2\displaystyle\int_{-a}^{a} f(x)\,\mathrm{d}x = 0 \implies \int_{-a}^{a} f(x)\,\mathrm{d}x = 0.$
The proof for f even is similar: substitute $u = -x$ for the integral $\int_0^a f(x)\,\mathrm{d}x$. ■

1. $(x^2-1)^2$ is even, hence
$$\int_{-1}^{1} (x^2-1)^2\,\mathrm{d}x = 2\int_0^1 (x^2-1)^2\,\mathrm{d}x = 2\int_0^1 x^4 - 2x^2 + 1\,\mathrm{d}x = 2\left(\frac{1}{5} - \frac{2}{3} + 1\right) = \frac{16}{15}$$
2. $\int_{-13\pi}^{13\pi} \sin(x^3 - 4x + \sin x)\,\mathrm{d}x = 0$
3. $\int_{-\pi/2}^{\pi/2} 4\sin x + \cos x\,\mathrm{d}x = 2\int_0^{\pi/2} \cos x\,\mathrm{d}x = 2\sin x\Big|_0^{\pi/2} = 2$

Suggested Problems

1. Evaluate the following integrals:

   (a) $\displaystyle\int s^{-1}\cos(\ln s)\,\mathrm{d}s$.

   (b) $\displaystyle\int_{-2}^{2} x\cos(x^2) + (2x-1)^2\,\mathrm{d}x$.

2. (a) Evaluate the integral $\displaystyle\int t\sin(t^2)\cos^8(t^2)\,\mathrm{d}t$.

   (b) Use the change of variables $u^3 = x^2 - 1$ to evaluate the integral $\displaystyle\int_1^3 x\sqrt[3]{x^2-1}\,\mathrm{d}x$.

3. (a) Evaluate the integral $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+\sqrt{1+x}}}$ (*start with* $u = \sqrt{1+x}\ldots$).

   (b) Let c be a non-zero constant. For any integrable function f, prove that

$$\int_a^b f(cx)\,\mathrm{d}x = \frac{1}{c}\int_{ac}^{bc} f(u)\,\mathrm{d}u.$$

Advanced: misusing the substitution rule

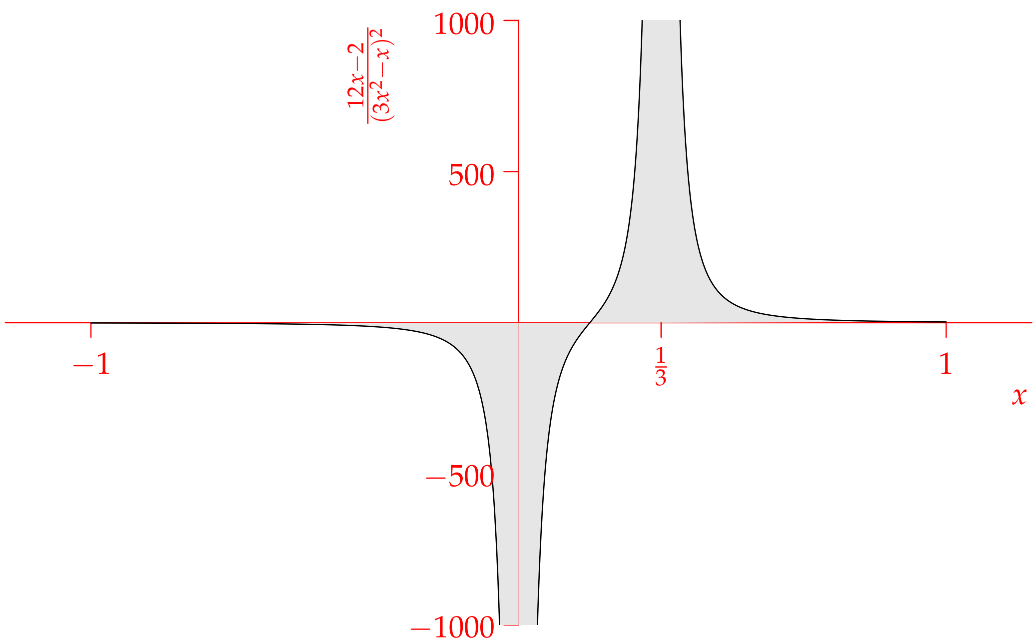
Suppose we wanted to evaluate $\displaystyle\int_1^3 \frac{12x-2}{(3x^2-x)^2}\,\mathrm{d}x$. We substitute $u = 3x^2 - x$ to get $\mathrm{d}u = (6x-1)\,\mathrm{d}x$:

$$\int_1^3 \frac{12x-2}{(3x^2-x)^2}\,\mathrm{d}x = \int_2^{24} \frac{2\,\mathrm{d}u}{u^2} = -\frac{2}{u}\bigg|_2^{24} = -\frac{1}{12} + 1 = \frac{11}{12}$$

There seems to be nothing wrong with this. Now try the same thing with different limits:

$$\int_{-1}^1 \frac{12x-2}{(3x^2-x)^2}\,\mathrm{d}x = \int_4^{2} \frac{2\,\mathrm{d}u}{u^2} = -\frac{2}{u}\bigg|_4^{2} = -1 + \frac{1}{2} = \frac{3}{2}$$

This second calculation is incorrect: why? One understand this immediately by thinking about the graph of the integrand $\frac{12x-2}{(3x^2-x)^2}$.



The integrand is discontinuous (indeed does not exist!) at both $x = 0$ and $x = \frac{1}{3}$, both of which are in the interval of integration $[-1, 1]$. We cannot apply the fundamental theorem of calculus to evaluate the integral. The area under the curve looks confusingly like $\infty - \infty$, which doesn't make sense. Indeed $\int_{-1}^1 \frac{12x-2}{(3x^2-x)^2}\,\mathrm{d}x = \text{DNE}$.

But why then did the subtitution rule transform a non-existant integral into something that we can evaluate correctly? $\frac{2}{u^2}$ is certainly continuous on the interval $[2, 4]$ so everything after the first equality is correct. The answer comes from a careful reading of the Substitution Rule Theorem.

- We wish to compute $\int_a^b f\big(g(x)\big)g'(x)\,\mathrm{d}x$ with $[a, b] = [-1, 1]$, $g(x) = 3x^2 - x$ and $f(u) = \frac{2}{u^2}$.
- Given the domain $[-1, 1]$, the range of $u = g(x)$ is the interval $[-\frac{1}{12}, 4]$.
- f is *not continuous* on this interval, so the hypotheses of the substitution rule are not satisfied.