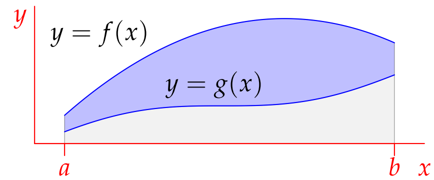
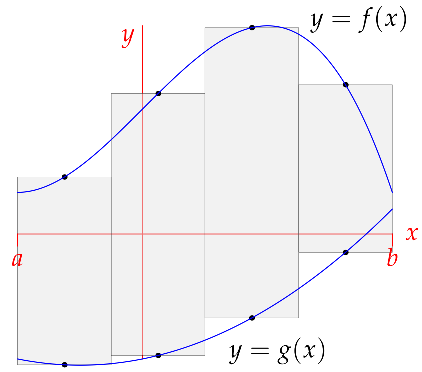
6 Applications of Integration

6.1 Areas between Curves



Computing the area between two curves is no harder than finding the area under each curve. If f, g are continuous and $f(x) \geq g(x) \geq 0$ as in the picture, then the area between the curves is the difference between the areas under f and g:

$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$



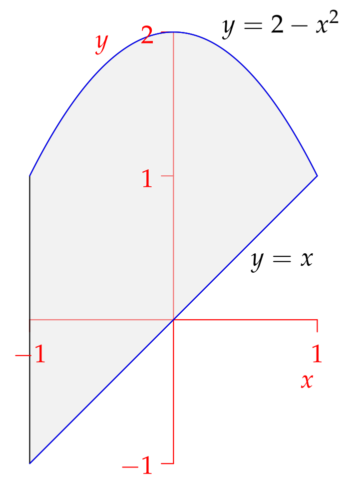
More generally we can find the area between curves $f(x) \geq g(x)$ by taking limits of a Riemann sum: note that the heights of the approximating rectangles are the difference between the values of f and g.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\Delta x = \int_a^b f(x) - g(x)\,dx$$

Examples

1. Find the area between $y = x$ and $y = 2 - x^2$ for $-1 \leq x \leq 1$.
   We simply compute the integral

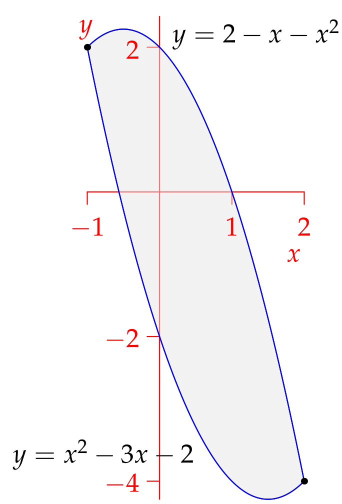
$$\int_{-1}^{1} 2 - x^2 - x\,dx = 2\int_0^1 2 - x^2\,dx$$
$$= 4x - \frac{2}{3}x^3\Big|_0^1$$
$$= 4 - \frac{2}{3} = \frac{10}{3}$$



   The first equality simplifies the integral by using the fact that x is odd and $2 - x^2$ is even.

2. Find the area between $y = 2 - x - x^2$ and $y = x^2 - 3x - 2$
   First we find the intersection points:

$$2 - x - x^2 = x^2 - 3x - 2$$
$$\Longrightarrow 0 = 2x^2 - 2x - 4 = 2(x-2)(x+1)$$
$$\Longrightarrow x = -1, 2$$

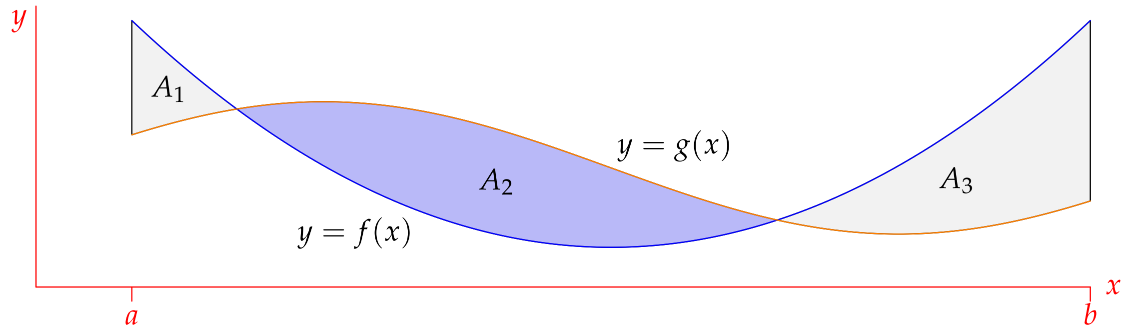


   The area is therefore

$$\int_{-1}^{2} 2 - x - x^2 - (x^2 - 3x - 2)\,dx = \int_{-1}^{2} 4 + 2x - 2x^2\,dx$$
$$= 4x + x^2 - \frac{2}{3}x^3\Big|_{-1}^{2}$$
$$= 12 + 3 - 6 = 9$$

Net Area

If $f(x) \ngeq g(x)$ then the integral $\int_a^b f(x) - g(x)\,dx$ calculates the *net area* between f and g. In the picture, the net area is $\int_a^b f(x) - g(x)\,dx = A_1 - A_2 + A_3$.



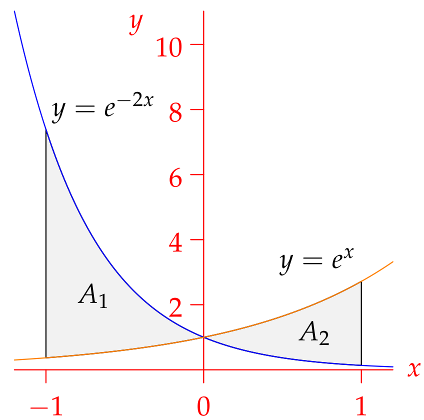
To find the area between the curves we must integrate the absolute value of the difference: in this case

$$\text{Total Area} = \int_a^b |f(x) - g(x)|\,dx = A_1 + A_2 + A_3$$

The challenge is to find the points of intersection of the curves and split the integral into several parts.

Example To find the total area between the curves $y = e^x$ and $y = e^{-2x}$ between $x = -1$ and $x = 1$, we first note that the curves meet at $x = 0$. The area is therefore

$$\begin{aligned} A_1 + A_2 &= \int_{-1}^{1} \left| e^{-2x} - e^x \right| \,dx \\ &= \int_{-1}^{0} e^{-2x} - e^x\,dx + \int_0^1 e^x - e^{-2x}\,dx \\ &= \frac{1}{-2}e^{-2x} - e^x \Big|_{-1}^{0} + e^x - \frac{1}{-2}e^{-2x}\Big|_0^1 \\ &= e + e^{-1} + \frac{1}{2}(e^2 + e^{-2}) - 3 \approx 3.84836 \end{aligned}$$



Integrating with respect to y

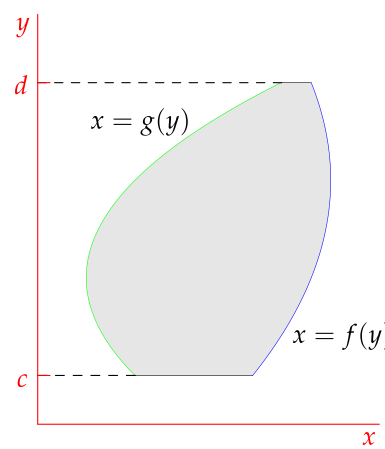
Sometimes it is simpler to regard a region as being bounded by functions of y instead of x. If a region is described by the inequalities

$$c \le y \le d, \quad g(y) \le x \le f(y)$$

then its net area is

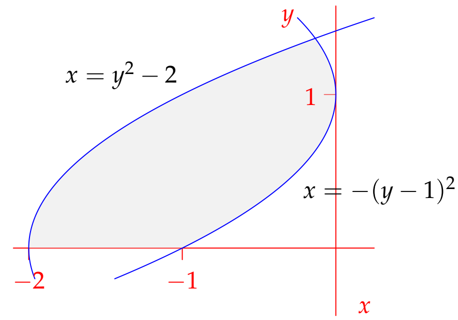
$$\int_c^d f(y) - g(y)\,dy$$

Flipping the picture in the line $y = x$ should convince you that this is correct.



Examples

1. Find the area shown.



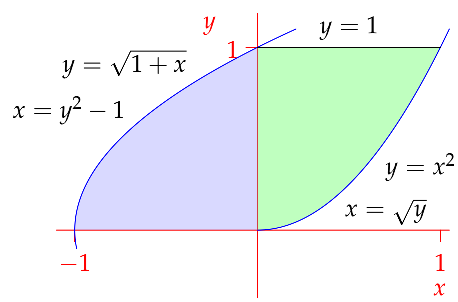
   The two curves meet when $y^2 - 2 = -(y-1)^2$ which is when

$$2y^2 - 2y - 1 = 0 \iff y = \frac{1 \pm \sqrt{3}}{2}$$

   The area is therefore

$$\int_0^{\frac{1+\sqrt{3}}{2}} -(y-1)^2 - (y^2-2)\,\mathrm{d}y = \int_0^{\frac{1+\sqrt{3}}{2}} -2y^2 + 2y + 1\,\mathrm{d}y$$
$$= -\frac{2}{3}y^3 + y^2 + y\Big|_0^{\frac{1+\sqrt{3}}{2}} = -\frac{2}{3}\left(\frac{1+\sqrt{3}}{2}\right)^3 + \left(\frac{1+\sqrt{3}}{2}\right)^2 + \frac{1+\sqrt{3}}{2}$$
$$= \frac{4+3\sqrt{3}}{6} \approx 1.5327$$

2. This area can be calculate in two ways.



   Calculate two x-integrals:

$$\text{Area} = \int_{-1}^{0} \sqrt{1+x}\,\mathrm{d}x + \int_0^1 1 - x^2\,\mathrm{d}x$$
$$= \frac{2}{3}(1+x)^{3/2}\Big|_{-1}^{0} + x - \frac{1}{3}x^3\Big|_0^1$$
$$= \frac{2}{3} + 1 - \frac{1}{3} = \frac{4}{3}$$

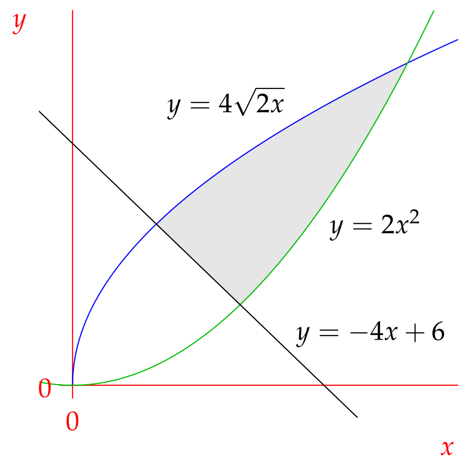
   Or a single y-integral:

$$\int_0^1 \sqrt{y} - (y^2 - 1)\,\mathrm{d}y = \frac{2}{3}y^{3/2} - \frac{1}{3}y^3 + y\Big|_0^1 = \frac{2}{3} - \frac{1}{3} + 1 = \frac{4}{3}$$

Suggested problems

1. Sketch the region bounded by the straight lines $y = x + 8$, $y = 8 - 2x$ and $y = 0$. Evaluate the area of the region using integral(s) with respect to x, and then with integral(s) with respect to y. Which approach was easier?

2. Find the area of the region sketched below.



3. *Without evaluating integrals*, explain why, for all positive integers n,

$$\int_0^1 x^n \, \mathrm{d}x + \int_0^1 \sqrt[n]{x} \, \mathrm{d}x = 1.$$