7 Techniques of Integration

7.1 Integration by Parts

The best that can be hoped for with integration is to take a rule from differentiation and reverse it. Integration by Parts is simply the Product Rule in reverse!

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)\\
\implies\ & f(x)g(x) = \int f'(x)g(x)\,\mathrm{d}x + \int f(x)g'(x)\,\mathrm{d}x\\
\implies\ & \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x
\end{aligned}$$

This can also be written

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

where $u = f(x), \quad \mathrm{d}u = f'(x)\,\mathrm{d}x, \quad v = g(x), \quad \mathrm{d}v = g'(x)\,\mathrm{d}x.$

Unfortunately, Integration by Parts is a lot less useful than the Product Rule to which it is equivalent.

- Applying it correctly means deciding which factor in the integrand is u and which ! $\mathrm{d}v$: these pieces must be treated differently. Moreover, you must know how to integrate the second factor before you start.
- The best it can achieve is to transform one integral into another. Hopefully this second integral is easier to compute.

Examples

1. Our first example is very simple. We recognize the integrand xe^x as a product of two functions and choose the simpler $u = x$ to be the function which we differentiate. The working is on the right.

$$\begin{aligned}
\int xe^x\,\mathrm{d}x &= xe^x - \int e^x\,\mathrm{d}x && (u = x, \quad \mathrm{d}v = e^x\,\mathrm{d}x)\\
&= xe^x - e^x + c && (\implies \mathrm{d}u = \mathrm{d}x, \quad v = e^x)
\end{aligned}$$

2. This time we take $u = 3x + 1$.

$$\begin{aligned}
\int (3x+1)\cos x\,\mathrm{d}x &= (3x+1)\sin x - \int 3\sin x\,\mathrm{d}x && (u = 3x+1, \quad \mathrm{d}v = \cos x\,\mathrm{d}x)\\
&= (3x+1)\sin x + 3\cos x + c && (\implies \mathrm{d}u = 3\,\mathrm{d}x, \quad v = \sin x)
\end{aligned}$$

Always write out the working for these; trying to compute with u and $\mathrm{d}v$ in your head is a recipe for disaster!

Which factor is u, which $\mathrm{d}v$?

1. If one factor is a polynomial, it is often easier to let this be u. Differentiating u results in a lower degree polynomial, whence the new integral $\int v\,\mathrm{d}u$ might be simpler than the original.
2. Can you integrate your choice of $\mathrm{d}v$? If not, may need to make u more complicated.

In our example above we calculated $\int xe^x\,\mathrm{d}x$. There are several possible choices for u and $\mathrm{d}v$. Here is what happens if you try them all:

u	$\mathrm{d}v$	$\mathrm{d}u$	v	$uv - \int v\,\mathrm{d}u$
e^x	$x\,\mathrm{d}x$	$e^x\,\mathrm{d}x$	$\frac{1}{2}x^2$	$\frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x\,\mathrm{d}x$
xe^x	$\mathrm{d}x$	$(1+x)e^x\,\mathrm{d}x$	x	$x^2e^x - \int x(1+x)e^x\,\mathrm{d}x$
x	$e^x\,\mathrm{d}x$	$\mathrm{d}x$	e^x	$xe^x - \int e^x\,\mathrm{d}x$

The first two choices in the above table result in a more difficult integral $\int v\,\mathrm{d}u$ than we started with. Only the final choice made things easier!

Can I choose any anti-derivative v of $\mathrm{d}v$? Notice how we didn't write $\mathrm{d}v = e^x\,\mathrm{d}x \implies v = e^x + c$. Why didn't we include a constant of integration. The reason is that it doesn't matter.

Lemma. *The choice of anti-derivative v of $\mathrm{d}v$ is irrelevant*

Proof. Suppose that $\hat{v}$ is another anti-derivative of $\mathrm{d}v$. Then $\hat{v} = v + c$ for some constant c, whence

$$u\hat{v} - \int \hat{v}\,\mathrm{d}u = u(v+c) - \int (v+c)\,\mathrm{d}u = uv + cu - \int v\,\mathrm{d}u - c\int \mathrm{d}u$$
$$= uv - \int v\,\mathrm{d}u + cu - cu = uv - \int v\,\mathrm{d}u$$

It follows that the choice of anti-derivative does not effect the Integration by Parts formula. ∎

Definite Integrals

We can easily adapt Integration by Parts to cope with definite integrals:

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x$$

Example Here we apply Integration by Parts twice, evaluating as we go.

$$\begin{aligned}
\int_0^\pi x^2\sin x\,\mathrm{d}x &= [-x^2\cos x]_0^\pi + \int_0^\pi 2x\cos x\,\mathrm{d}x && (u = x^2,\ \mathrm{d}v = \sin x\,\mathrm{d}x)\\
&= -\pi^2\cos\pi + \int_0^\pi 2x\cos x\,\mathrm{d}x && (\implies \mathrm{d}u = 2x,\ v = -\cos x)\\
&= -\pi^2\cos\pi + [2x\sin x]_0^\pi - \int_0^\pi 2\sin x\,\mathrm{d}x && (\tilde{u} = 2x,\ \mathrm{d}\tilde{v} = \cos x\,\mathrm{d}x)\\
&= \pi^2 + 2\cos x\big|_0^\pi && (\implies \mathrm{d}\tilde{u} = 2,\ \tilde{v} = \sin x)\\
&= \pi^2 - 4
\end{aligned}$$

Try the same thing to see that $\int_0^\pi x^4\cos x\,\mathrm{d}x = 4\pi(6-\pi^2)$.

Recurrence Formulæ

Complicated integrals can often be simplified using multiple applications of the technique. For example, when faced with

$$\int e^{-2x} \cos 3x \, \mathrm{d}x$$

we don't know which factor to choose: exponentials and sines/cosines don't qualitatively change whether you integrate or differentiate. We therefore don't expect Integration by Parts to help us, but let's try it anyway.

If we let $u = e^{-2x}$ and $\mathrm{d}v = \cos 3x \, \mathrm{d}x$, then $\mathrm{d}u = -2e^{-2x} \, \mathrm{d}x$ and $v = \frac{1}{3} \sin 3x$. Plugging this into the formula, we obtain

$$\int e^{-2x} \cos 3x \, \mathrm{d}x = \frac{1}{3} e^{-2x} \sin 3x + \frac{2}{3} \int e^{-2x} \sin 3x \, \mathrm{d}x$$

The result is an integral that looks almost the same as what we started with! Rather than give up, we try the technique again. Let $\tilde{u} = e^{-2x}$ and $\mathrm{d}\tilde{v} = \sin 3x \, \mathrm{d}x$, then $\mathrm{d}\tilde{u} = -2e^{-2x} \, \mathrm{d}x$ and $v = -\frac{1}{3} \cos 3x$. Thus

$$\int e^{-2x} \cos 3x \, \mathrm{d}x = \frac{1}{3} e^{-2x} \sin 3x + \frac{2}{3} \left(-\frac{1}{3} e^{-2x} \cos 3x - \frac{2}{3} \int e^{-2x} \cos 3x \, \mathrm{d}x \right)$$

This almost looks useless, until you realize that we've returned to the integral we started with. Indeed if you use the letter I to represent the original integral, then the above is an algebraic equation for I which can be easily solved:

$$\begin{aligned} & I = \frac{1}{3} e^{-2x} \sin 3x - \frac{2}{9} e^{-2x} \cos 3x - \frac{4}{9} I \\ \implies & \frac{13}{9} I = \frac{1}{3} e^{-2x} \sin 3x - \frac{2}{9} e^{-2x} \cos 3x \\ \implies & 13I = 3e^{-2x} \sin 3x - 2e^{-2x} \cos 3x \end{aligned}$$

It follows that our original integral is

$$\int e^{-2x} \cos 3x \, \mathrm{d}x = \frac{1}{13} e^{-2x} (3 \sin 3x - 2 \cos 3x) + c$$

A similar approach allows us to calculate sequences of integrals. For instance, if n is a positive integer, we can let $I_n = \int x^n e^x \, \mathrm{d}x$. A single application (try it!) of integration by parts shows that

$$I_n = x^n e^x - n I_{n-1}$$

By iterating this expression, we can quickly compute larger integrals: for example

$$\begin{aligned} \int x^3 e^x \, \mathrm{d}x &= I_3 = x^3 e^x - 3I_2 = x^3 e^x - 3(x^2 e^x - 2I_1) = x^3 e^x - 3x^2 e^x + 6I_1 \\ &= x^3 e^x - 3x^2 e^x + 6xe^x - 6I_0 = (x^3 - 3x^2 + 6x - 6)e^x + c \end{aligned}$$

since $I_0 = \int e^x \, \mathrm{d}x = e^x + c$.

Unexpected Applications

Sometimes taking $\mathrm{d}v = \mathrm{d}x$ gives surprising results. This approach only works in very special situations!

Examples

1. Let $u = \ln x$ and $\mathrm{d}v = \mathrm{d}x$, then $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = x$

$$\int \ln x \,\mathrm{d}x = \ln x \cdot x - \int x \cdot \frac{1}{x}\,\mathrm{d}x = x\ln x - \int \mathrm{d}x = x(\ln x - 1) + c$$

2. Similarly $u = \sin^{-1}x \implies \mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$

$$\int \sin^{-1}x \,\mathrm{d}x = \sin^{-1}x \cdot x - \int x \cdot \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = x\sin^{-1}x + \sqrt{1-x^2} + c$$

Suggested problems

1. Evaluate the following integrals:
   (a) $\int te^{3t}\,\mathrm{d}t$
   (b) $\int_0^{1/2} 2\sin^{-1}(2t)\,\mathrm{d}t$ $\qquad (\sin^{-1}(2t) = \arcsin(2t))$
   (c) $\int e^{\sqrt{x}}\,\mathrm{d}x$ $\qquad$ (try a substitution first)
2. (a) Make the substitution $u = \sin^{-1}x$ and then integrate by parts:

$$\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}}\,\mathrm{d}x$$

   (b) i. If $I_n = \int_0^1 x^n e^{-x}\,\mathrm{d}x$, prove the recurrence relation

$$I_n = nI_{n-1} - e^{-1}$$

   ii. Hence, or otherwise, evaluate the integral

$$\int_0^1 x^3 e^{-x}\,\mathrm{d}x$$

3. Consider the function $f(x) = e^{-x}\sin x$ for $x \geq 0$.
   (a) Sketch the function for $0 \leq x \leq 4\pi$.
   (b) If f is rotated around the x-axis, the result is an infinite string of beads of decreasing volume. Find the volume of the first two beads.
   (c) Find the volume of entire infinite string of beads (you will need to sum an infinite series, see chapter 11...)