7.3 Trigonometric Substitution

These are useful for integrating square-roots of quadratic expressions. That is, if your intergand contains any terms of the form $\sqrt{ax^2+bx+c}$, where a, b, c are constant.

We have already seen an example using the substitution $x = \sin\theta$:

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,\mathrm{d}\theta = \int \frac{\cos\theta}{\cos\theta}\,\mathrm{d}\theta = \int \mathrm{d}\theta = \theta + c = \sin^{-1} x + c$$

General Strategies

There are three primary types of expression. Each can be simplified by a trigonometric substitution.

1. If the integrand contains $\sqrt{a^2-x^2}$ let $x = a\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then

   $$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\cos\theta \quad \text{and} \quad \mathrm{d}x = a\cos\theta\,\mathrm{d}\theta$$

   We now have an integral containing sines and cosines, which is (hopefully) amenable to the methods of the last section.

2. If the integrand contains $\sqrt{a^2+x^2}$ let $x = a\tan\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then

   $$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sec\theta \quad \text{and} \quad \mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$$

   We now have an integral containing secants and tangents.

3. If the integrand contains $\sqrt{x^2-a^2}$ let $x = a\sec\theta$ where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, then

   $$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta \quad \text{and} \quad \mathrm{d}x = a\sec\theta\tan\theta\,\mathrm{d}\theta$$

   We again have an integral containing secants and tangents.

Examples

1. For our first example of the method, we check an integral that may be easily dispatched via the susbtitution $u = 4 - x^2$.

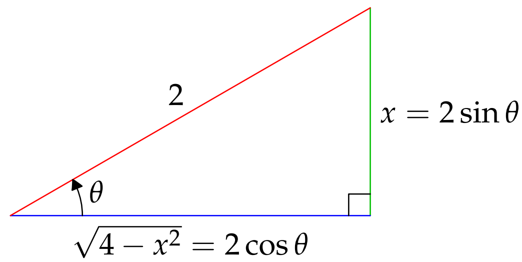
   $$\int x\sqrt{4-x^2}\,\mathrm{d}x = \int \sqrt{u}(-\tfrac{1}{2}\,\mathrm{d}u) = -\frac{1}{3}u^{3/2} + c = -\frac{1}{3}(4-x^2)^{3/2} + c$$

   Instead we apply the methods of this section: let $x = 2\sin\theta$, then $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$, and

   $$\begin{aligned}\int x\sqrt{4-x^2}\,\mathrm{d}x &= \int 2\sin\theta\sqrt{4+4\sin^2\theta}\cdot 2\cos\theta\,\mathrm{d}\theta = 8\int \sin\theta\cdot\cos^2\theta\,\mathrm{d}\theta \\ &= -\frac{8}{3}\cos^3\theta + c \qquad \text{(substitute } u = \cos\theta \text{ explicitly if you need to)} \\ &= -\frac{8}{3}\left(\cos(\sin^{-1}\tfrac{x}{2})\right)^3 + c\end{aligned}$$

This answer is revolting! How can we simplify the $\cos(\sin^{-1})$ expression? Since trigonometric functions are defined using right-angled triangles, we draw one with angle $\theta = \sin^{-1}\frac{x}{2}$. This says that

$$\frac{x}{2} = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$



so we draw our triangle with opposite x and hypotenuse 2. We want

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

which, after applying Pythagoras' Theorem to calculate the length $\sqrt{4-x^2}$ of the adjacent, gives us our result:

$$\int x\sqrt{4-x^2}\,\mathrm{d}x = -\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + c = -\frac{1}{3}(4-x^2)^{3/2} + c$$

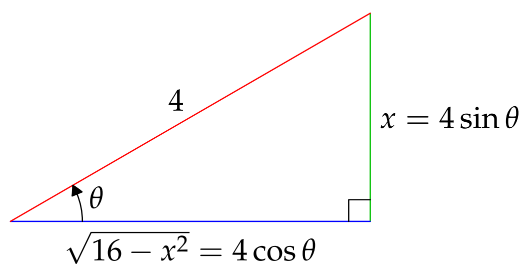
as before.

2. This time we use $x = 4\sin\theta$: then $\mathrm{d}x = 4\cos\theta\,\mathrm{d}\theta$, and

$$\int \frac{1}{(16-x^2)^{3/2}}\,\mathrm{d}x = \int \frac{1}{(16-16\sin^2\theta)^{3/2}}\cdot 4\cos\theta\,\mathrm{d}\theta = \int \frac{4\cos\theta}{16^{3/2}(\cos^2\theta)^{3/2}}\,\mathrm{d}\theta$$
$$= \frac{1}{16}\int \sec^2\theta\,\mathrm{d}\theta = \frac{1}{16}\tan\theta + c = \frac{x}{16\sqrt{16-x^2}} + c$$

To finish things off we needed another triangle, drawn below.

$$\frac{x}{4} = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$



so we draw our triangle with opposite x and hypotenuse 4. We want

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

3. For definite integrals, we can change the limits as we go, so no triangle pictures are necessary. Here we let $x = \sqrt{2}\sin\theta$ then $\mathrm{d}x = \sqrt{2}\cos\theta\,\mathrm{d}\theta$. The limits become $x = 0 \iff \theta = 0$ and $x = \sqrt{2} \iff \theta = \frac{\pi}{2}$, whence

$$\int_0^{\sqrt{2}} x^3\sqrt{2-x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} 2\sqrt{2}\sin^3\theta\sqrt{2-2\sin^2\theta}\cdot\sqrt{2}\cos\theta\,\mathrm{d}\theta$$
$$= 4\sqrt{2}\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,\mathrm{d}\theta = 4\sqrt{2}\int_0^{\frac{\pi}{2}} (\cos^2\theta - \cos^4\theta)\sin\theta\,\mathrm{d}\theta$$
$$= 4\sqrt{2}\left(-\frac{1}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{8\sqrt{2}}{15}$$

Example could alternatively have been done via the substitution $u = 2 - x^2$: try it!

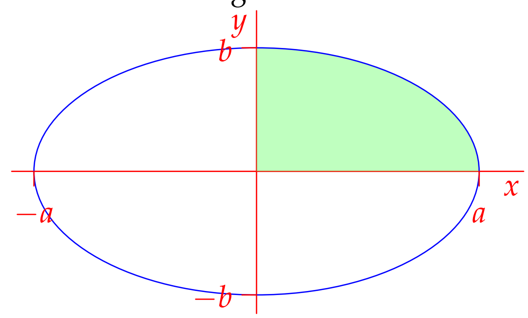
The Area of an Ellipse

An ellipse with *semi-major axis a* and *semi-minor axis b* has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Its total area is four times the area of the upper-right quadrant: this is the integral

$$A = 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,\mathrm{d}x = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x$$

which can be computed using the substitution $x = a\sin\theta$. Remember to change the limits...

$$\begin{aligned} A &= \frac{4b}{a}\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta\,\mathrm{d}\theta \\ &= 4ab\int_0^{\pi/2} \cos^2\theta\,\mathrm{d}\theta \\ &= 2ab\int_0^{\pi/2} 1 + \cos 2\theta\,\mathrm{d}\theta = \pi ab \end{aligned}$$



Examples with Secant and Tangent Substitutions

1. Let $x = 3\sec\theta$ to obtain $\mathrm{d}x = 3\sec\theta\tan\theta\,\mathrm{d}\theta$. Then

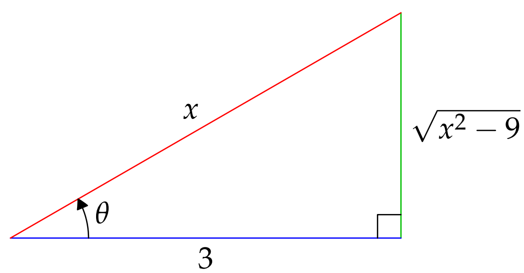
$$\begin{aligned} \int \frac{\sqrt{x^2 - 9}}{x}\,\mathrm{d}x &= \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta}\cdot 3\sec\theta\tan\theta\,\mathrm{d}\theta \\ &= 3\int \tan^2\theta\,\mathrm{d}\theta = 3\int \sec^2\theta - 1\,\mathrm{d}\theta \\ &= 3\tan\theta - 3\theta + c \\ &= \sqrt{x^2 - 9} - 3\sec^{-1}\frac{x}{3} + c \end{aligned}$$

The last step requires a triangle.

$$\frac{x}{3} = \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

so we draw our triangle with hypotenuse x and adjacent 3. We want

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{x^2 - 9}}{3}$$



2. This time we set $x = 5\tan\theta$:

$$\begin{aligned} \int \frac{1}{x^2\sqrt{25 + x^2}}\,\mathrm{d}x &= \int \frac{5\sec^2\theta}{25\tan^2\theta\cdot 5\sec\theta}\,\mathrm{d}\theta = \frac{1}{25}\int \frac{1}{\tan^2\theta\cos\theta}\,\mathrm{d}\theta = \frac{1}{25}\int \frac{\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta \\ &= -\frac{1}{25\sin\theta} + c = -\frac{\sqrt{25 + x^2}}{25x} + c \end{aligned}$$

Try drawing the required triangle yourself.

More general expressions $\sqrt{Q(x)}$

By completing the square and changing variables, any quadratic $Q(x)$ may be transformed to one of the standard forms.

Example By completing the square, $6x - x^2 = 9 - (x-3)^2$ which, through the substitution $u = x - 3$, yields

$$\begin{aligned}\int \frac{x}{\sqrt{6x - x^2}}\,\mathrm{d}x &= \int \frac{x}{\sqrt{9-(x-3)^2}}\,\mathrm{d}x = \int \frac{3+u}{\sqrt{9-u^2}}\,\mathrm{d}u = 3\sin^{-1}\frac{u}{3} - \sqrt{9-u^2} + c \\ &= 3\sin^{-1}\left(\frac{x-3}{3}\right) - \sqrt{6x-x^2} + c\end{aligned}$$

Suggested problems

1. (a) Evaluate the integral $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x$

   (b) Evaluate $\int \frac{x\,\mathrm{d}x}{\sqrt{16+4x^2}}$ using a trigonometric substitution. What method would have been easier?

2. Consider the function $f(x) = (9+x^2)^{-1/2}$ on the interval $[0,4]$

   (a) Find the area under the curve $y = f(x)$.

   (b) Find the volume when the region under the curve is rotated around the x-axis.

   (c) (Hard) Find the volume when the region under the curve is rotated around the y-axis.

3. Evaluate the integral $\int \frac{x^2+2x+4}{\sqrt{x^2-4x}}\,\mathrm{d}x$, $x > 4$ (*You may quote the integrals of* $\sec\theta$ *and* $\sec^3\theta$)