7.5 Strategy for Integration

Integration is *hard* relative to differentiation. Almost any function built from elementary functions can be differentiated using the Product, Quotient, and Chain Rules. With integration you have essentially three tools:

1. A list of standard forms (the master-list below is more than enough for this course)
2. Substitutions
3. Integration by parts

It is entirely possible that you are faced with an integral that cannot be computed using these methods. For example

$$\int \sin(\cos x)\,\mathrm{d}x \qquad \text{and} \qquad \int \frac{1}{\sqrt{1+2x^2+3x^3}}\,\mathrm{d}x$$

cannot be evaluated using the methods of this class[1]

Standard Integration Formulæ

Be absolutely sure you know the first column. The second column is less important, but still useful.

$$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} \quad (n \neq -1)$$
$$\int x^{-1}\,\mathrm{d}x = \ln|x|$$
$$\int \sin x\,\mathrm{d}x = -\cos x$$
$$\int \cos x\,\mathrm{d}x = \sin x$$
$$\int \sec^2 x\,\mathrm{d}x = \tan x$$
$$\int e^x\,\mathrm{d}x = e^x$$
$$\int a^x\,\mathrm{d}x = \frac{1}{\ln a}a^x$$
$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$
$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} \quad (a>0)$$

$$\int \csc^2 x\,\mathrm{d}x = -\cot x$$
$$\int \sec x \tan x\,\mathrm{d}x = \sec x$$
$$\int \csc x \cot x\,\mathrm{d}x = -\csc x$$
$$\int \sec x\,\mathrm{d}x = \ln|\sec x + \tan x|$$
$$\int \csc x\,\mathrm{d}x = \ln|\csc x - \cot x|$$
$$\int \tan x\,\mathrm{d}x = \ln|\sec x|$$
$$\int \cot x\,\mathrm{d}x = \ln|\sin x|$$
$$\int \frac{\mathrm{d}x}{x^2-a^2} = \ln\left|\frac{x-a}{x+a}\right|$$
$$\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln\left|x+\sqrt{x^2 \pm a^2}\right|$$

Further Steps

Here is a four step strategy for how to approach an integral when it is not in the standard table

1. Can the integrand be simplified? Expressions like $(x+3x^{3/2})^2$ are better multiplied out

$$\int (x+3x^{3/2})^2\,\mathrm{d}x = \int x^2+6x^{5/2}+9x^3\,\mathrm{d}x = \frac{1}{3}x^3+\frac{12}{7}x^{7/2}+\frac{9}{4}x^4+c$$

[1] Power series, and other, methods can be used to find approximations

Similarly $\frac{\csc^2 x}{\tan x} = \frac{\cos x}{\sin^3 x}$ is easier to integrate in the second form:

$$\int \frac{\cos x}{\sin^3 x} \, \mathrm{d}x = -\frac{1}{2\sin^2 x} + c$$

2. Is there an obvious/easy substitution? For example $u = x^3 - x^2 + 1$ with differential $\mathrm{d}u = (3x^2 - 2x)\,\mathrm{d}u$ is the obvious choice in

$$\int \frac{2x - 3x^2}{\sqrt{x^3 - x^2 + 1}} \, \mathrm{d}x = \int \frac{-\,\mathrm{d}u}{\sqrt{u}} = -2\sqrt{u} + c = -2\sqrt{x^3 - x^2 + 1} + c$$

   The integral $\int (3x^2 - x^{-2})(x^3 + x^{-1})\,\mathrm{d}x$ can either be multiplied out, or evaluated using a simple substitution: it should *not* be attempted by parts!

   - Multiplying out:

$$\int (3x^2 - x^{-2})(x^3 + x^{-1})\,\mathrm{d}x = \int 3x^5 + 2x - x^{-3}\,\mathrm{d}x = \frac{1}{2}x^6 + x^2 + \frac{1}{2}x^{-2} + c$$

   - Substitution: Let $u = x^3 + x^{-1}$, then

$$\int (3x^2 - x^{-2})(x^3 + x^{-1})\,\mathrm{d}x = \int u\,\mathrm{d}u = \frac{1}{2}u^2 + c = \frac{1}{2}(x^3 + x^{-1})^2 + c$$

3. The big guns. Classify the integrand according to its form

   - Trigonometric functions: convert everything to combinations of sin/cos or sec/tan and use one of the Trigonometric Integrals methods.
   - Rational functions: use the method of partial fractions.
   - Integration by Parts: if the integrand is a product, especially where one factor is a power of x, consider integration by parts.
   - Radicals: If the integrand contains square-roots of quadratic terms $\sqrt{Q(x)}$, try a trigonometric substitution.
     If the integrand has a root of a linear expression $\sqrt[n]{ax+b}$ try substituting $u^n = ax + b$ which might simplify the integrand.

4. Try again! Try another substitution, a more creative use of integration by parts, even try guessing something and manipulating your guess based on its derivative.

The message of the above is twofold:

- Try something simple *before* resorting to any of the tough methods (partial fractions, parts, etc.).
- Integration is an art form: what you try might not work, so be willing to try something else and experiment!