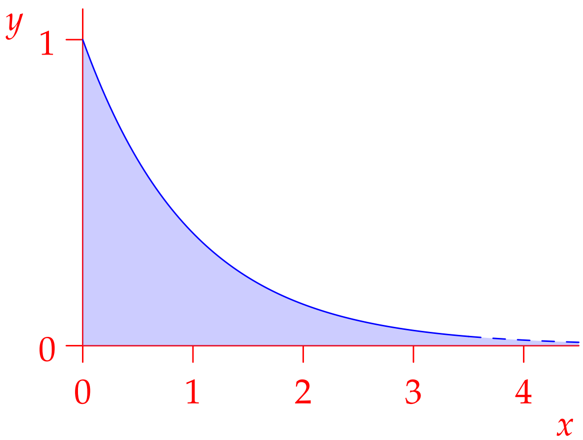
7.8 Improper Integrals

The Riemann Sum definition of an integral has problems if the endpoints a, b are infinite or if f is undefined at $x = a$ or $x = b$.

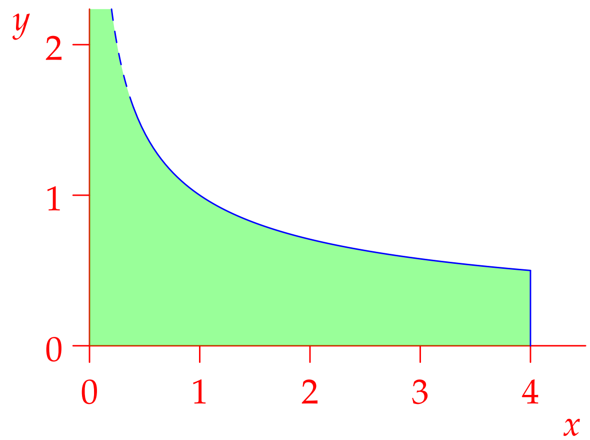
Definition. An integral $\int_a^b f(x)\,\mathrm{d}x$ is *improper* in either of the following cases:

Type 1 $a = -\infty$ and/or $b = \infty$: we write $\int_{-\infty}^b f(x)\,\mathrm{d}x$ or $\int_a^\infty f(x)\,\mathrm{d}x$.

Type 2 f is discontinuous at $x = a$ and/or b.



Type I: $\displaystyle\int_0^\infty e^{-x}\,\mathrm{d}x$



Type II: $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

An integral could be Type 1 at one end and Type 2 at the other, e.g. $\displaystyle\int_1^\infty \frac{1}{(x-1)^2}\,\mathrm{d}x$

Convergent and divergent integrals

Just because we can write down an integral does not mean that it exists! The existence or otherwise of improper integrals relies on the existence of certain limits. For our example of a Type I integral above, it is clear that for any *finite* value t, we have

$$\int_0^t e^{-x}\,\mathrm{d}x = -e^{-x}\big|_0^t = 1 - e^{-t}$$

It seems reasonable that if $\int_0^\infty e^{-x}\,\mathrm{d}x$ is to make any sense, then it should be equal to the limit:

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \lim_{t\to\infty}\int_0^t e^{-x}\,\mathrm{d}x = \lim_{t\to\infty} 1 - e^{-t} = 1$$

Indeed this is our definition, and we say that $\int_0^\infty e^{-x}\,\mathrm{d}x$ is a *convergent improper integral* with value 1. Colloquially, it might be said that the blue region in the above picture has 'area' equal to 1, though this is no notion of area with which you are familiar: can a region really have *finite* area when its *length* (extension along the x-axis) is infinite?
Similarly,

$$\lim_{t\to 0^+}\int_t^4 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{t\to 0^+}\left[4 - 2t^{1/2}\right] = 4 \implies \int_0^4 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 4$$

which is another convergent improper integral.

Definition. 1. Suppose that $\int_a^t f(x)\,\mathrm{d}x$ exists and is finite for every $t \in [a, b)$ (where b could be ∞). Then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x$$

If the limit exists and is finite, we say that the improper integral is *convergent*. Otherwise the integral is *divergent*.

2. Similarly (here a could be $-\infty$),

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{t\to a^+} \int_t^b f(x)\,\mathrm{d}x$$

3. Finally, suppose that $\int_a^b f(x)\,\mathrm{d}x$ is an improper integral at *both* a and b. Let c be some value in the interval (a, b). We say that $\int_a^b f(x)\,\mathrm{d}x$ is convergent if and only if *both* the integrals $\int_c^b f(x)\,\mathrm{d}x$ and $\int_a^c f(x)\,\mathrm{d}x$ are convergent, in which case

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

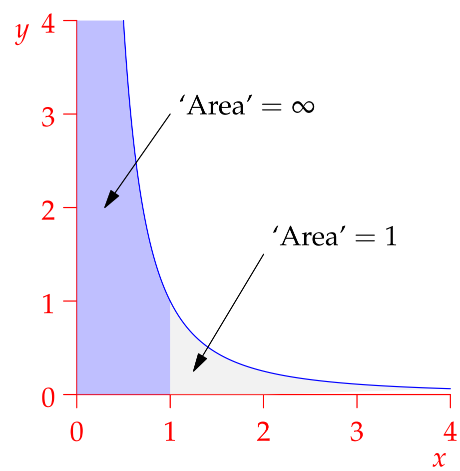
4. The expression $\int_a^b f(x)\,\mathrm{d}x = \infty$ is often used to mean

$$\lim_{t\to b^-} \int_a^t f(x)\,\mathrm{d}x = \infty$$

that is, for an improper integral which diverges to infinity.

Example

1. $\displaystyle\lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,\mathrm{d}x = \lim_{t\to\infty} 1 - \frac{1}{t} = 1 \implies \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1$
2. $\displaystyle\lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,\mathrm{d}x = \lim_{t\to 0^+} \frac{1}{t} - 1 = \infty \implies \int_0^1 \frac{1}{x^2}\,\mathrm{d}x = \infty$



Therefore $\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x$ is convergent and $\int_0^1 \frac{1}{x^2}\,\mathrm{d}x$ is divergent. This is an example of a general result.

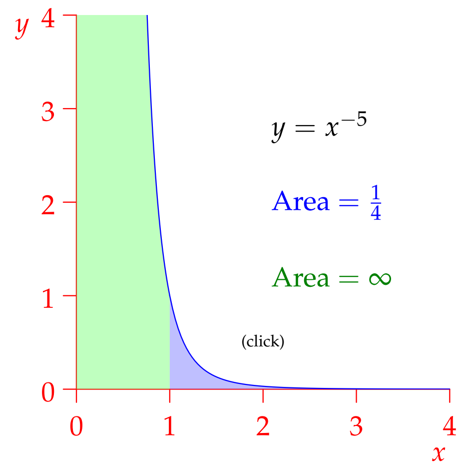
Theorem. *1.* $\displaystyle\int_1^\infty \frac{1}{x^p}\,\mathrm{d}x$ *is convergent* $\iff p > 1$.

2. $\displaystyle\int_0^1 \frac{1}{x^p}\,\mathrm{d}x$ *is convergent* $\iff p < 1$.

To re-iterate, the integrals $\displaystyle\int_0^1 \frac{1}{x}\,\mathrm{d}x$ and $\displaystyle\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ are *both* divergent.

Proof. For part 1,

$$\begin{aligned}
\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\,dx \\
&= \begin{cases} \lim\limits_{t\to\infty} \frac{1}{1-p} x^{1-p}\Big|_1^t & \text{if } p \neq 1 \\ \lim\limits_{t\to\infty} \ln x\Big|_1^t & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \frac{1}{p-1}\left[1 - \lim\limits_{t\to\infty} t^{1-p}\right] & \text{if } p \neq 1 \\ \lim\limits_{t\to\infty} \ln t & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \le 1 \end{cases}
\end{aligned}$$

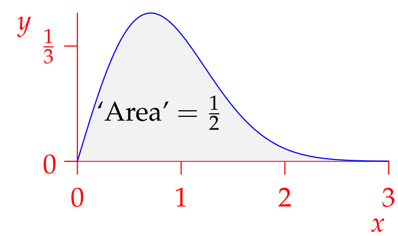


Part 2. is similar. ■

Examples To speed up calculations, we can write

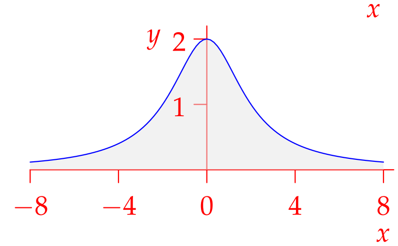
$$\int_a^\infty f(x)\,dx = F(x)\Big|_a^{\to\infty} = \lim_{t\to\infty} F(t) - F(a)$$

1. $$\int_0^\infty xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_0^{\to\infty} = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.$$



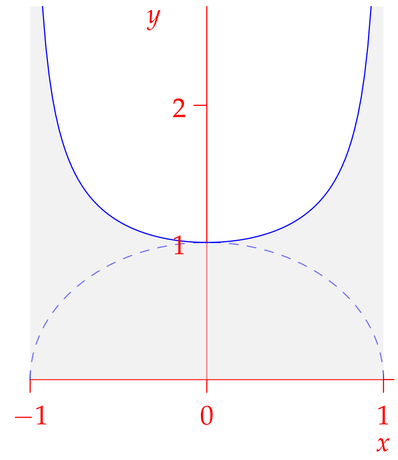
2. This integral is improper at both ends.

$$\begin{aligned}
\int_{-\infty}^\infty \frac{8}{4+x^2}\,dx &= \frac{8}{2}\tan^{-1}\frac{x}{2}\Big|_{\to-\infty}^{\to\infty} \\
&= 4\left[\lim_{t\to\infty}\tan^{-1}\frac{t}{2} - \lim_{s\to-\infty}\tan^{-1}\frac{s}{2}\right] \\
&= 4\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 4\pi
\end{aligned}$$



3. Another integral which is improper at both ends.

$$\begin{aligned}
\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx &= \sin^{-1} x\Big|_{\to-1}^{\to1} \\
&= \lim_{t\to1^+}\sin^{-1} t - \lim_{s\to1^-}\sin^{-1} s \\
&= 2\sin^{-1} 1 = \pi
\end{aligned}$$



The Comparison Test for Integrals: non-examinable

It is useful to be able to compare integrals: if we can compare a complicated integral to one which is understood, we can often decide if the complicated integral converges or diverges.

Theorem. *Suppose that f and g are continuous functions on (a,b) which satisfy $0 \le f(x) \le g(x)$. Then*

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$$

It follows that

1. *If $\int_a^b g(x)\,\mathrm{d}x$ is convergent, then $\int_a^b f(x)\,\mathrm{d}x$ is convergent*
2. *If $\int_a^b f(x)\,\mathrm{d}x$ is divergent, then $\int_a^b g(x)\,\mathrm{d}x$ is divergent*

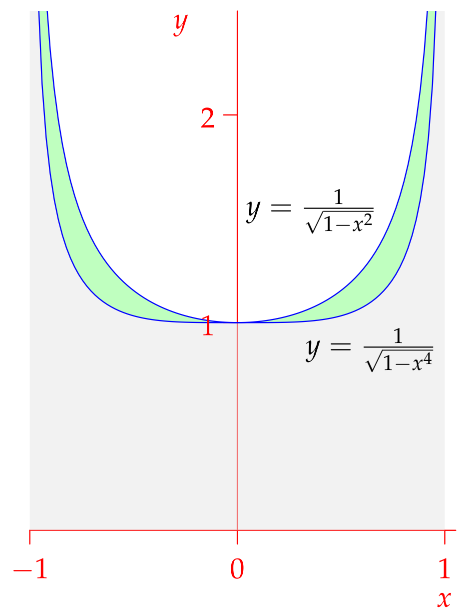
Example The improper integral $\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\,\mathrm{d}x$ cannot be computed precisely using techniques from this course. However,

$$\begin{aligned} x \in (-1,1) &\implies x^2 \ge x^4 \\ &\implies \sqrt{1-x^2} \le \sqrt{1-x^4} \\ &\implies \frac{1}{\sqrt{1-x^4}} \le \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

It follows that

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\,\mathrm{d}x \le \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \pi$$

whence we conclude that the integral $\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\,\mathrm{d}x$ converges.
In fact $\int_{-1}^{1} \frac{1}{\sqrt{1-x^4}}\,\mathrm{d}x \approx 2.622$.



Suggested problems

1. (a) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{1}{(1+2x)^2}\,\mathrm{d}x$
   (b) i. Calculate the derivative of $\sqrt{x}e^{-x}$.
       ii. Hence or otherwise, evaluate $\displaystyle\int_0^{\infty} \frac{1-2x}{\sqrt{x}e^x}\,\mathrm{d}x$
2. (a) Evaluate the integral: $\displaystyle\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2}\,\mathrm{d}x$
   (b) (Harder) Let $f(x) = \begin{cases} x^{-1/n} = \frac{1}{\sqrt[n]{x}} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$
   For what positive integers n does the integral $\int_{-1}^{1} f(x)\,\mathrm{d}x$ make sense? Evaluate it when it does. (*Hint: first check when the integrand is defined for $x < 0$…*)

3. *Gabriel's Horn* is obtained by rotating the curve $y = \frac{1}{x}$ around the x-axis, for $1 \leq x < \infty$.

   (a) Calculate the volume of the horn.

   (b) The surface area of a surface of revolution obtained by rotating $y = f(x)$ around the x-axis for $a \leq x \leq b$ is given by the integral

   $$\int_a^b f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

   Compute the surface area of Gabriel's Horn.

   (c) Investigate the 'Painter's Paradox' on the web. See if you can make sense of it!