MATH 150 HOMEWORK 3

DUE: Monday, Oct 30
Student Name/ID # (Include all students in the group):

IMPORTANT INSTRUCTION: It is crucial that you clearly explain how you arrive at your conclusions. The point of the homework is to practice understanding of the material and the ability to express your understanding.

1. (**5 pts**) A set Σ of formulas is independent iff for every formula $\phi \in \Sigma$,

$$\Sigma \backslash \{\phi\} \not\models \phi.$$

Here $\Sigma \backslash \{\phi\}$ is the set of formulas obtained by removing the formula ϕ from Σ. (More generally, if X, Y are sets then $X \backslash Y$ is the set obtained by removing all elements of Y from X.) Of course, if $\phi \notin \Sigma$, then $\Sigma \backslash \{\phi\} = \Sigma$.

(a) (**1 pt**) Is the set of formulas $\Sigma = \{A \to C, A \vee B, \neg B, C\}$ independent? Prove or disprove.

(b) (**0.5pt+0.5pt**) Let Σ and Σ' be sets of formulas such that $\Sigma' \subseteq \Sigma$. Prove or disprove. (Try to understand what is going on.)

(i) For every formula σ we have: If $\Sigma' \models \sigma$ then $\Sigma \models \sigma$.

(ii) For every formula σ we have: If $\Sigma \models \sigma$ then $\Sigma' \models \sigma$.

(c) (**2 pts**) Given a finite set $\Sigma = \{\phi_1, \phi_2, \ldots, \phi_n\}$ of formulas, write a recipe how to pick a subset $\Sigma' \subseteq \Sigma$ such that Σ' is independent and $\Sigma' \models \Sigma$.

(d) (**1pt**) Let $\Sigma' \subseteq \Sigma$ be sets of formulas such that $\Sigma' \models \phi$ for all $\phi \in \Sigma$. Prove or disprove: For every formula σ, if $\Sigma \models \sigma$ then $\Sigma' \models \sigma$.

2. (**5 pts**) Let Σ be a set of formulas.

(a) (**2 pts**) Assume Σ is satisfiable. Show that if ϕ is any formula then one of the sets $\Sigma \cup \{\phi\}, \Sigma \cup \{\neg\phi\}$ is satisfiable.

(b) (**1 pt**) Prove that Σ is not satisfiable if and only if $\Sigma \models A \wedge \neg A$.

(c) (**2 pts**) Assume Σ is such that for every formula ϕ,

$$(\star) \quad \text{If } \Sigma \models \phi \text{ then } \Delta \models \phi \text{ for some finite } \Delta \subseteq \Sigma.$$

Prove: If every finite $\Delta \subseteq \Sigma$ is satisfiable then Σ is satisfiable. Do not use the Compactness Theorem in your proof. This gives you a proof of the Compactness Theorem from the property $(\star)$. **Hint.** Use (b).

3.(**5 pts**) Consider the following situation. You have a collection of infinitely many cubes $C_0, C_1, C_2, C_3, \ldots$. Some of them are connected with wire. Apply the Compactness Theorem to the following tasks.

(a) (**3 pts**) Assume you have two colors, and whenever you pick a finite subcollection of cubes, it is possible to color them using just those two colors so that no two cubes connected with a wire have the same color. Prove that the entire infinite collection can be colored in the same manner, that is, you use just those two colors and no two cubes connected with a wire get the same color.

(b) (**2 pts**) Do the analogous task as in (a), but now use four colors.

Hint. (a): View C_i as sentential symbols, and colors $0, 1$ as truth evaluations. Write down formulas in those sentential symbols that express that two cubes connected with a wire get different colors. Subtle point: Do not express by a formula in your formal language that two cubes are connected with a wire. Instead, use the information about which are connected to write the correct set of formulas. (b): Add sentential letters D_i and code each color by evaluations of the pair C_i, D_i (note that there are four possible evaluations, which correspond to the number of colors).