MATH 150 HOMEWORK 4 SOLUTION

1. (**5pts**) For each of the following structures find a language for the respective structure and describe the interpretations of symbols of your language. Recall that for each function and relational symbol you need to specify the number of arguments.

(a) **(1pt)** The domain of the structure is the set of all integers $\mathbb{Z}$. There are two specified objects in the domain, namely 0 and 1. The structure has usual operations of addition and multiplication. Additionally, there are the following functions: exponentiation $n \mapsto 2n$, the function which to each two integers assigns their distance, and a function that to each tuple of integers of length five assigns the largest element.

(b) **(2pts)** The domain of the structure is the set of all real numbers R. For each integer $n > 0$ the structure has the function that to each finite tuple of real numbers of length n assigns the mean value. Additionally, for each integer $n > 0$ the structure has the relation that for each finite tuple of real numbers of length n tells whether the distances between the adjacent numbers are all the same.

(c) **(2pts)** The domain of the structure is the set of all lines in the Euclidean plane. The structure has the relation "line l is parallel with line l'". Additionally, the structure has relations that for each triple of lines carry the information about how many lines in the triple are parallel – make this precise.

Proof. (a) The language $\mathcal{L}_1$ of this structure has the following (non-logical) symbols:

(i) $\dot{0}, \dot{1}$: constant symbols. These are interpreted as $0, 1$ respectively in the structure.

(ii) $\dot{+}, \dot{\times}, \dot{E}, \dot{d}$: 2-ary function symbols. These are interpreted as $+, \times$, exponentiation, distance functions in the structure.

(iii) $m\dot{a}x$: 5-ary function symbol. $m\dot{a}x$ is interpreted as the function taking in tuples of length 5 and outputs the largest element of the tuple.

(b) The language $\mathcal{L}_2$ has the following (non-logical) symbols:

(i) For each n, F_n is an n-ary function symbol. F_n is interpreted in the structure as a function whose input is a tuple of length n and whose output is the mean of the tuple.

(ii) For each n, R_n is an n-ary relation symbol. R_n is interpreted as the n-ary relation on the real numbers that tells whether the distances between the adjacent numbers are all the same.

(c) The language $\mathcal{L}_3$ has the following (non-logical) symbols:

(i) R: 2-ary relation symbol. R is interpreted in the structure as a 2-ary relation R^* such that given any two lines l, l', $R^*(l, l')$ is true iff l is parallel to l'.

(ii) For $n \in \{0, 1, 2, 3\}$, S_n: 3-ary relation symbol. S_n is interpreted in the structure as a 3-ary relation S_n^* such that given any three lines l, l', l'', $S_n^*(l, l', l'')$ is true iff there are exactly n pairs of lines that are parallel. For example, $S_1(l, l', l'')$ is true iff there is exactly one pair (out of the three possible pairs) of lines that is parallel.

□

2. (**5 pts**) Given is a structure $\mathfrak{M}$ characterized as follows. The domain M of the structure $\mathfrak{M}$ is the set of all people. The language has one constant symbol p, two unary function symbols F and G and one binary function symbol C. The interpretation of the constant symbol p is:

$$p^{\mathfrak{M}} = \text{the president of the people in } M.$$

The interpretations $F^{\mathfrak{M}}, G^{\mathfrak{M}}$, and $C^{\mathfrak{M}}$ of these symbols are two unary functions

$$F^{\mathfrak{M}} : M \to M \text{ and } G^{\mathfrak{M}} : M \to M$$

and a binary function

$$C^{\mathfrak{M}} : M \times M \to M$$

with the following interpretations: For each $m \in M$,

$$F^{\mathfrak{M}}(m) = \text{the mother of } m,$$
$$G^{\mathfrak{M}}(m) = \text{ the father of } m,$$

and for each pair $(m, n) \in M \times M$, $C^{\mathfrak{M}}(m, n) =$

- the oldest child of the couple (m, n) if (m, n) has a child;
- the oldest child amongst all children of m, n if the couple (m, n) doesn't have a child but at least one of m, n does have a child;
- the president if m, n do not have any children or (m, n) is not a couple.

For each of the following situations write down the term describing the following persons.

(a) **(1pt)** The great-grandmother of u from the father's and grandfather's side.

(b) **(1pt)** The grandfather from the mother's side of the grandmother from the mother's side of u.

(c) **(1pt)** The oldest sibling of u, granting that u has a sibling.

(d) **(1pt)** The oldest uncle/aunt of u from mother's side, granting u has one.

(e) **(1pt)** The husband of the president's daughter, granting that the president has only one child, this child is a daughter, and she has a child with her husband.

Proof. I suggest you draw the ancestral trees to make it easy to visualize these relations. I will omit the superscript $\mathfrak{M}$ in the most of the following terms to make things less cluttered.

(a) $F(G(G(u)))$ (technically, $F^{\mathfrak{M}}(G^{\mathfrak{M}}(G^{\mathfrak{M}}(u)))$).

(b) $G(F(F(F(u))))$.

(c) $C(F(u), G(u))$ (note here that the oldest sibling of u may be u; yes, I know this is not what "oldest sibling" typically means. But in our situation, we are provided with only the function C, so we do not have "enough" functions to separate u from another sibling in case u is the oldest child).

(d) $C(F(F(u)), G(F(u))$ (again for the same reason as (c), the oldest uncle/aunt of u from the mother's side may be u's mother).

(e) I do not see how to write this term without adding another function: $H^{\mathfrak{M}} : M \to M$, where $H^{\mathfrak{M}}(m) =$ the spouse of m if m has a spouse, and $p^{\mathfrak{M}}$ otherwise. Apologies, I will not grade this part on your homework.

First, the president's daughter can be expressed as $C(p, H(p))$ (assuming here relevant people are not divorced). The husband of the president's daughter can be expressed as: $H(C(p, H(p)))$.

□

3. (**5pt**) Let $\mathcal{L} = \{\dot{0}, \dot{S}, \dot{+}, \dot{\times}, \dot{<}\}$ be the language of number theory introduced in class. Express each of the following statements as a formula (or sentence) in $\mathcal{L}$.

(a) **(1pt)** "Numbers u, v are consecutive primes".

(b) **(1pt)** "Numbers u, v are relative primes".

(c) **(1pt)** "w is the greatest common divisor of u, v"

(d) **(1pt)** "Numbers u, v, w, z constitute an arithmetic sequence"

(e) **(1pt)** "There are infinitely many primes"

Recall that numbers a, b are relative primes if they have only one common divisor, namely number 1. Also, d is the greatest common divisor of a, b iff d is a common divisor of a, b and every common divisor of a, b divides d. Also recall that an increasing sequence of numbers $(a_1, a_2, \ldots, a_k)$ is arithmetic if the distances between all consecutive numbers in the sequence are all equal.

Proof. I will just write the final statements here. First have a formula $\varphi(u) \equiv \dot{<}(\dot{S}(\dot{0}), u) \wedge \forall v(v \dot{<} u \wedge \dot{S}(\dot{0}) \dot{<} v \to \neg(\exists w(\dot{\times}(v, w) \dot{=} u))$. $\varphi(u)$ just says that "u is prime".

Now we make another formula $\psi(u, v) \equiv \exists w(\dot{\times}(u, w) \dot{=} v)$. $\psi(u, v)$ just says "u divides v".

(a) $\phi(u) \wedge \phi(v) \wedge \dot{<}(u, v) \wedge \forall w(u \dot{<} w \wedge w \dot{<} v \to \neg\phi(w))$

(b) Our formula is: $\forall w(\dot{S}(\dot{0}) \dot{<} w \to \neg(\psi(w, u) \wedge \psi(w, v)))$.

(c) $\psi(w, u) \wedge \psi(w, v) \wedge \forall y(\psi(y, u) \wedge \psi(y, v) \to \neg(w \dot{<} y))$.

(d) $u \dot{<} v \wedge v \dot{<} w \wedge w \dot{<} z \wedge \exists d(\dot{+}(u, d) \dot{=} v \wedge \dot{+}(v, d) \dot{=} w \wedge \dot{+}(w, d) \dot{=} z)$.

(e) You can say "for any natural number n, there is a $p > n$ such that p is prime". This is equivalent to the statement in (e). So our formula is: $\forall u \exists v(u \dot{<} v \wedge \varphi(v))$.

□