A COHERENT SEQUENCE IN $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})$

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1. INTRODUCTION

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation (here $\mathcal{P}$ is countable). We work under the assumption $\mathsf{AD}^+ + \Theta = \Theta_\Sigma + \mathsf{MC}(\Sigma)$. Under this assumption, $V = \mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})$, where $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})$ is the stack of countably iterable Θ-g-organized premice over $\mathbb{R}$ that project to $\mathbb{R}$ as defined in [4]. A proof of this fact can be found in [1]. The construction of the coherent sequence here is heavily based on the construction of $\square_\kappa$ sequences in $L[E]$ models done in [3], though much complexity that appears in [3] due to the existence of pluripotent $L[E]$ levels that gives rise to protomice does not occur in our situation.

Before stating our main result, let us fix some notation. Let $C = \{\tau < \Theta \mid \mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})|\tau \vDash \mathsf{ZF}^-$ and there are no $\mathbb{R}$-cardinals in $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})|\tau\}$.[1] Notice that C is a club in Θ by reflection and the fact that $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})|\Theta \vDash \mathsf{ZF}^-$. For each $\tau \in C$, let N_τ be the least segment of $\mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})$ such that there is an $n < \omega$ such that $\rho_n^{N_\tau} \geq \tau$ and $\rho_{n+1}^{N_\tau} = \mathbb{R}$. Notice that $o(N_\tau) > \tau$ and $\tau = \Theta^{N_\tau}$. Furthermore, τ is a strong cutpoint of N_τ, that is τ is not measurable in N_τ and there are no extenders on the sequence of N_τ that overlap τ.

Now, we're ready to state our main result.

Theorem 1.1. *There is a sequence $\vec{D} = \langle D_\tau \mid \tau \in C\rangle$ with the following properties:*

1. $D_\tau \subseteq \tau \cap C$
2. *D_τ is closed, and if $cof(\tau) > \omega$ then D_τ is unbounded in τ.*
3. *if $\overline{\tau} < \tau$, then $D_{\overline{\tau}} = D_\tau \cap \overline{\tau}$.*

We say E is a *thread through* $\vec{D}$ if $E \subseteq C$ is club in Θ and for every limit point α of E, $E \cap \alpha = D_\alpha$. We remark that by the construction, if Θ has uncountable cofinality in a parent universe with the same reals, then the sequence $\vec{D}$ does not admit a thread since any thread through $\vec{D}$ will produce an $\mathcal{M} \lhd \mathrm{Lp}^{\mathsf{G}\Sigma}(\mathbb{R})$ such that $\rho_\omega^{\mathcal{M}} = \mathbb{R}$ and $o(\mathcal{M}) \geq \Theta$. Contradiction.

[1] Here ZF^- is ZF minus Powerset Axiom.

We note that by the same proof, we obtain the same result if we assumed $V = \mathrm{Lp}^{g\Sigma}(\mathbb{R})$, i.e. V is the stack of countably iterable g-organized premice over $\mathbb{R}$ that project to $\mathbb{R}$ (see [4]).

We remark that the construction below goes through for both Jensen indexing and Mitchell-Steel indexing. In the construction of [3], Jensen indexing is used since it relies on the appropriate condensation lemma for that indexing. Here, we simply replace instances where the condensation lemma is used by a simple comparison argument, which does not depend on any particular indexing scheme.

Finally, we remark that the sequence $\vec{D}$ can be turned into a coherent sequence $\vec{D}'$ on Θ in the sense of [2] by purely combinatorial means. The sequence $\vec{D}'$ is similarly nonthreadable in the sense above.

2. THE CONSTRUCTION

We fix some more notations. Here are some objects associated to each $\tau \in C$. The notations follow closely those of [3].

- $\langle S_\alpha^{E,{}^{\mathsf{G}}\Sigma}(\mathbb{R}) \mid \alpha < \Theta\rangle$, $\langle J_\alpha^{E,{}^{\mathsf{G}}\Sigma}(\mathbb{R}) \mid \alpha < \Theta\rangle$ are the Jensen's S and J hierarchies relative to ${}^{\mathsf{G}}\Sigma$ and E, which is the extender sequence of $\mathrm{Lp}^{{}^{\mathsf{G}}\Sigma}(\mathbb{R})$.
- n_τ is the unique n such that $\rho_n^{N_\tau} \geq \tau$ and $\rho_{n+1}^{N_\tau} = \mathbb{R}$.
- p_τ is the standard parameter for N_τ.
- $\rho_\tau = \rho_{n_\tau}^{N_\tau}$.
- $\mathcal{H}_\tau$ is the n_τ-th reduct of N_τ.
- $\tilde{h}_\tau = \tilde{h}_{N_\tau}^{n_\tau+1}$ is the $\Sigma_1^{(n_\tau)}$ Skolem function of N_τ, so there is a set $\tilde{H}_\tau$ such that $y = \tilde{h}_\tau(x, p_\tau)$ where $x \in \mathbb{R}$ if and only if $\exists z \in \mathcal{H}_\tau, (z, y, x, p_\tau) \in \tilde{H}_\tau$.

For more details and background on fine structure, see [3]. Our construction here resembles that of the square sequence in [3] for the $\langle C_\alpha \mid \alpha \in \mathcal{S}^0\rangle$, where $\mathcal{S}^0$ is, roughly speaking, the set over which the Jensen's construction of square in L can be adapted in the $L[E]$ context in a straightforward manner. Here the set $\mathbb{S}^0$ in [3] is C.

We first define an approximation $\langle B_\tau \mid \tau \in C\rangle$ to our desired sequence $\langle D_\tau \mid \tau \in C\rangle$.

Definition 2.1. *Let $\tau \in C$, B_τ is the set of all $\overline{\tau} \in C \cap \tau$ satisfying:*

- $n_\tau = n_{\overline{\tau}} = n$, *and* $N_{\overline{\tau}}, N_\tau$ *are* Θ-g-*organized* $\mathbb{R}$-*premice of the same type.*[2]

[2] In Mitchell-Steel indexing, premice are of types I, II, III; in Jensen indexing, premice are of types A, B, C.

- *There is a map $\sigma_{\overline{\tau}\tau}: N_{\overline{\tau}} \to N_\tau$ which is $\Sigma_0^{(n_\tau)}$-preserving (with respect to the language of Θ-g-organized $\mathbb{R}$-premice) and such that*

  1. *$\overline{\tau} = cr(\sigma_{\overline{\tau}\tau})$, and $\sigma_{\overline{\tau}\tau}(\overline{\tau}) = \tau$;*
  2. *$\sigma_{\overline{\tau}\tau}(p_{\overline{\tau}}) = p_\tau$;*
  3. *for each $\alpha \in p_\tau$, there is a generalized witness $Q_\tau(\alpha)$ for α with respect to N_τ and p_τ such that $Q_\tau(\alpha) \in rng(\sigma_{\overline{\tau}\tau})$.*

For a discussion on solidity witnesses and generalized witnesses, see [5] or [3]. The important point is that being a generalized witness (being a $\Pi_1^{(n_\tau)}$ notion) is preserved downward under the embeddings $\sigma_{\overline{\tau}\tau}$.

Notice also that the maps $\sigma_{\overline{\tau}\tau}$ are uniquely determined (and the uniqueness of such maps is not influenced by (3) in the definition). To see this, let $a \in N_{\overline{\tau}}$, then $\exists x \in \mathbb{R}, a = \tilde{h}_{\overline{\tau}}(x, p_{\overline{\tau}})$; this $\Sigma_1^{(n)}$ definition is preserved upwards by the $\Sigma_0^{(n)}$ map $\sigma_{\overline{\tau}\tau}$ and as $\sigma_{\bar{\tau}\tau}(x) = x$ for any real x, $\sigma_{\overline{\tau}\tau}(a) = \tilde{h}_\tau(x, p_\tau)$.

Lastly, the maps $\sigma_{\overline{\tau}\tau}$ are not cofinal at the n-th level, and hence not $\Sigma_1^{(n)}$ preserving. Otherwise, by soundness of the N_τ's, the maps $\sigma_{\overline{\tau}\tau}$ are onto, hence $N_{\overline{\tau}} = N_\tau$, which is not possible.

Lemma 2.2. *Let $\tau \in C$, and $\tau^* < \overline{\tau}$ in B_τ. Then $rng(\sigma_{\tau^*\tau}) \subseteq rng(\sigma_{\overline{\tau}\tau})$.*

Proof. First we show

$$sup((\sigma_{\tau^*\tau})''\omega\rho_{\tau^*}) < sup((\sigma_{\overline{\tau}\tau})''\omega\rho_{\overline{\tau}}) \tag{2.1}$$

Suppose this is false. Pick an $x \in \mathbb{R}$ such that $\tilde{h}_{\overline{\tau}}(x, p_{\overline{\tau}}) = \tau^*$, so there is a $z \in \mathcal{H}_{\overline{\tau}}$ such that $\tilde{H}_{\overline{\tau}}(z, \tau^*, x, p_{\overline{\tau}})$. Applying $\sigma_{\overline{\tau}\tau}$, we get $\tilde{H}_\tau(\sigma_{\overline{\tau}\tau}(z), \tau^*, x, p_\tau)$. Choose $\overline{\zeta} < \omega\rho_{\overline{\tau}}$ such that $z \in S_{\overline{\zeta}}^{E,{}^\mathfrak{G}\Sigma}(\mathbb{R})$ and let $\zeta = \sigma_{\overline{\tau}\tau}(\overline{\zeta})$. By the failure of 2.1, there is a $\zeta^* < \omega\rho_{\tau^*}$ such that $\zeta \leq \zeta' = \sigma_{\tau^*\tau}(\zeta^*)$. So the statement $(\exists u^n \in S_{\zeta'}^{E,{}^\mathfrak{G}\Sigma}(\mathbb{R}))(\exists\delta^n < \tau)\tilde{H}_\tau(u^n, \delta^n, x, p_\tau)$ is a $\Sigma_0^{(n)}$ formula held in N_τ, hence can be pulled back by $\sigma_{\tau^*\tau}$. So $h_{\tau^*}(x, p_{\tau^*})$ is defined and $\sigma_{\tau^*\overline{\tau}}(h_{\tau^*}(x, p_{\tau^*})) = \tau^*$, which contradicts the fact that $\tau^* = cr(\sigma_{\tau^*\tau})$. This proves 2.1.

2.1 can be used to show that $\forall x \in \mathbb{R}, \tilde{h}_{\tau^*}(x, p_{\tau^*})$ is defined $\longrightarrow$ $\tilde{h}_{\overline{\tau}}(x, p_{\overline{\tau}})$ is defined. To see this, we need to see that the relation $(\exists x^0)(x^0 = \tilde{h}_\tau(x, p_\tau))$ is uniformly $\Sigma_1^{(n)}$. This follows from the fact that $\tilde{h}_\tau$ is a good $\Sigma_1^{(n)}$-function, so substituting $\tilde{h}_\tau$ for v^0 in the relation $(\exists x^0)(x^0 = v^0)$ yields the result. The rest is just as in the previous paragraph.

From the discussion in the previous paragraph, we know $\sigma_{\tau^*\tau}(\tilde{h}_{\tau^*}(x, p_{\tau^*})) = \sigma_{\overline{\tau}\tau}(\tilde{h}_{\overline{\tau}}(x, p_{\overline{\tau}}))$ whenever $\tilde{h}_{\tau^*}(x, p_{\tau^*})$ is defined. By soundness of N_{τ^*}, we have the desired conclusion. □

Remark 2.3. *In the proof of the above lemma,* (3) *of Definition 2.1 is never used.*

Lemma 2.4. *Let $\tau \in C$ and $\overline{\tau} \in B_\tau$. Then $B_\tau \cap \overline{\tau} = B_{\overline{\tau}} - min(B_\tau)$.*

Proof. Let $\tau^* \in B_\tau \cap \overline{\tau}$. We'll show that $\tau^* \in B_{\overline{\tau}}$. By the previous lemma, $rng(\sigma_{\tau^*\tau}) \subseteq rng(\sigma_{\overline{\tau}\tau})$, we can define the map $\sigma : N_{\tau^*} \to N_{\overline{\tau}}$ by $\sigma = (\sigma_{\overline{\tau}\tau})^{-1} \circ \sigma_{\tau^*\tau}$. It's easy to see that σ satisfies all requirements in Definition 2.1 except maybe for item 3. But this is also true because we can pull back the generalized witness by a $\Sigma_0^{(n)}$ map. Hence $\sigma = \sigma_{\tau^*\overline{\tau}}$ and $\tau^* \in B_{\overline{\tau}}$.

Let $\tau' = min(B_\tau)$ and $\tau^* \in B_{\overline{\tau}} - \tau'$. We may assume $\tau^* > \tau'$. Define $\sigma = \sigma_{\overline{\tau}\tau} \circ \sigma_{\tau^*\overline{\tau}}$. To show that $\sigma = \sigma_{\tau^*\tau}$, it suffices to verify (3) of Definition 2.1 as the other conditions are evident. If $Q(\alpha) \in rng(\sigma_{\tau'\tau})$ is a generalized witness for $\alpha \in p_\tau$ with respect to N_τ and p_τ then $Q(\alpha) \in rng(\sigma)$ by the previous lemma, so $\tau^* \in B_\tau$. □

The above lemma tells us that the sequence $\langle B_\tau \mid \tau \in C \rangle$ is almost coherent. Inspecting the proof of the lemma, it's easy to see that condition (3) in the Definition 2.1 is the reason full coherency may fail. We now modify the sequence $\langle B_\tau \mid \tau \in C \rangle$ a bit to get full coherency. For $\tau \in C$, we let

- $\tau(0) = \tau$
- $\tau(i+1) = min(B_{\tau(i)})$
- $l_\tau =$ the least i such that $B_{\tau(i)} = \varnothing$

Notice that for any $\tau \in C$, l_τ is defined and is less than ω. Now for any $\tau \in C$,

- $D_\tau = B_{\tau(0)} \cup B_{\tau(1)} \cup ... \cup B_{\tau(l_\tau - 1)}$, and
- for any $\tau^* \in D_\tau$, $\sigma_{\tau^*\tau} = \sigma_{\tau(1)\tau(0)} \circ \sigma_{\tau(2)\tau(1)} \circ ... \circ \sigma_{\tau^*\tau(j)}$ where j is such that $\tau^* \in B_{\tau(j)}$. Note that the maps $\sigma_{\tau^*\tau}$ are unique with critical point τ^*, sending τ^* to τ, and p_{τ^*} to p_τ.

Lemma 2.5. *The sequence $\langle D_\tau \mid \tau \in C \rangle$ is coherent.*

Proof. Let $\tau \in C$ and $\tau^* \in D_\tau$. First it is easy to see that

$$min(B_{\tau^*}) \in D_\tau,\ B_{\tau^*} = D_\tau \cap [min(B_{\tau^*}), \tau^*) \tag{2.2}$$

Now define $\tau^*(i)$ from τ^* the same way $\tau(i)$ is defined from τ. Using (2.2), it is easy to see that D_{τ^*} must be an initial segment of D_τ. □

It remains to prove for every $\tau \in C$, D_τ is closed and if $cof(\tau) > \omega$, D_τ is unbounded.

Lemma 2.6. *Suppose $\tau \in C$ and $cof(\tau) > \omega$. Then D_τ is unbounded in τ.*

Proof. We first assume N_τ is E-active, that is, $N_\tau = \langle J_\alpha^{E,^{\mathcal{G}}\Sigma}(\mathbb{R}), F_\tau \rangle$, where $F_\tau \neq \emptyset$ is the (amenable code) for the top extender of N_τ. Say $cr(F_\tau) = \kappa$ and $\vartheta = (\kappa^+)^{N_\tau}$. Note that $\kappa > \tau$; this is because F_τ is $\mathbb{R}$-complete over N_τ and $\tau = \Theta^{N_\tau}$, as discussed above, is a strong cutpoint for N_τ. We may assume $n = 1$, where $\omega\rho_{n+1}^{N_\tau} = \mathbb{R}$, and $\omega\rho_n^{N_\tau} \geq \tau$. Let $\tau^* < \tau$. We want to show there is a $\tilde{\tau} \in D_\tau$ such that $\tilde{\tau} \geq \tau^*$.

First, let $X = \tilde{h}_\tau(\{\tau^*, \mathcal{P}\} \cup \{p_\tau\})$. Then X is countable Σ_1 substructure of N_τ. Let $\sigma : \overline{N} \to N_\tau$ be the inverse of the collapse and $\sigma(\overline{\kappa}, \overline{\vartheta}, \overline{\tau}, \overline{p}, \overline{\kappa}) = (\kappa, \vartheta, \tau, p_\tau, \kappa)$. Note that $\overline{\tau} < \overline{\kappa}$ and $\overline{\vartheta} = (\overline{\kappa}^+)^{\overline{N}}$.

Now let $\tilde{\tau} = sup(\sigma''\overline{\tau})$. Then $\tau^* < \tilde{\tau}$ because we put τ^* in X, and $\tilde{\tau} < \tau$ because X is countable and $cof(\tau) > \omega$. We claim that $\tilde{\tau} \in D_\tau$.

By [4, Lemma 3.10], $\overline{N}$ is Θ-g-organized. Suppose $\overline{N} = \langle J_{\overline{\alpha}}^{\overline{E},^{\mathcal{G}}\Sigma}(\overline{\mathbb{R}}), \overline{F} \rangle$. Then $\overline{N}$ is a $\overline{\mathbb{R}}$-premouse of the same type as N_τ as an $\mathbb{R}$-premouse. Let $\tilde{N}$ be the ultrapower of $\overline{N}$ by the extender induced by σ with generators in $\mathbb{R}$ and the ultrapower is formed using functions $f \in \overline{N}$; so $\tilde{N}$ is isomorphic to $\{\sigma(f)(x) \mid f \in \overline{N} \wedge x \in \mathbb{R}\}$. Let $\tilde{\sigma}$ be the ultrapower map and $\sigma' : \tilde{N} \to N_\tau$ be the natural map. So $\sigma = \sigma' \circ \tilde{\sigma}$. It's not hard to see that Los theorem holds and σ' is Σ_0 preserving, hence $\tilde{N}$ is wellfounded. From now on, we identify $\tilde{N}$ with its transitive collapse.

Now, $\tilde{N}$ contains all the reals. We need to see that $\tilde{N}$ is a Θ-g-organized $\mathbb{R}$-premouse of the same type as N_τ. The fact that $\tilde{N}$ is Θ-g-organized follows from [4, Lemma 3.10] and the fact that σ' embeds $\tilde{N}$ into the Θ-g-organized $\mathbb{R}$-premouse N_τ. Now $\tilde{\sigma}$ is Σ_0 and cofinal and $cr(\bar{F}) \geq \bar{\tau} = cr(\tilde{\sigma})$. This implies the top extender of $\tilde{N}$ measures all sets in $\tilde{N}$ and hence $\tilde{N}$ is a premouse (and not a protomouse).

Let $x \in \tilde{N}$; so $x = \tilde{\sigma}(f)(a)$ for some $f \in \overline{N}$ and $a \in \mathbb{R}$. But $f = \tilde{h}_{\overline{N}}(b, \overline{p})$ for some $b \in \mathbb{R}$, so $\tilde{\sigma}(f) = h_{\tilde{N}}(b, \tilde{p})$, where $\tilde{p} = \tilde{\sigma}(\overline{p}) = \sigma'^{-1}(p_\tau)$. So $x = \tilde{h}_{\tilde{N}}(b, \tilde{p})(a)$. This gives $\omega\rho_{\tilde{N}}^{n+1} = \mathbb{R}$, $\tilde{\tau} \leq \omega\rho_n^{\tilde{N}}$, and $\tilde{p} \in R_{\tilde{N}}$. Solidity of $\tilde{N}$ is guaranteed by the fact that $\sigma' : \tilde{N} \to N_\tau$ is $\Sigma_0^{(n)}$-preserving. Since $rng(\sigma) \subseteq rng(\sigma')$, $rng(\sigma')$ contains a generalized witness for $\sigma'(\beta)$ for each $\beta \in \tilde{p}$. This shows $\tilde{p}$ is the standard parameter for $\tilde{N}$ and $\tilde{N}$ is sound.

Finally, we show $\tilde{N} = N_{\tilde{\tau}}$ by showing that $\tilde{N} \lhd N_\tau$. Note that $\tilde{N} \neq N_\tau$ because $\tilde{\tau}$ is a cardinal in $\tilde{N}$ but $\tilde{\tau} < \tau$ and hence cannot be a cardinal in N_τ. $\tilde{N}$ is countably iterable since it is embeddable into N_τ. Hence $\tilde{N} \lhd N_\tau$ by a simple comparison argument using the fact that both are sound, Θ-g-organized premice over $\mathbb{R}$ projecting to $\mathbb{R}$.[3]

[3] In [3], this is where the index-dependent condensation lemma is used to show the corresponding fact about levels of $L[E]$. In our case, things are greatly simplified as $\mathbb{R}$ is a "strong cutpoint" of both $\tilde{N}$ and N_τ; hence, we can show $\tilde{N} \lhd N_\tau$ by directly comparing them. The argument that neither side moves during the comparison is the same as that in the proof that two sound mice projecting to ω are lined up.

Now suppose N_τ is either B-active (that is, the top predicate of N_τ codes a branch) or passive. Let $\bar{N}, \tilde{N}, \tilde{\sigma}, \sigma', \tilde{\tau}$ be defined as above. Again, by [4, Lemma 3.10], $\tilde{N}$ is a countably iterable Θ-g-organized, $\mathbb{R}$-sound mouse over $\mathbb{R}$. The same argument as above shows $\tilde{N} = N_{\tilde{\tau}}$. This completes the proof of the lemma.

□

Lemma 2.7. *Suppose $\tau \in C$. Then D_τ is closed.*

Proof. Let $\tilde{\tau}$ be a limit point of D_τ. We need to see that $\tilde{\tau} \in D_\tau$. First, note that $\tilde{\tau} \in C$ as C is closed. Now form the direct limit $\langle \tilde{N}, \sigma_{\tau^*\tilde{\tau}} \mid \tau^* \in D_\tau \rangle$ of the direct system $\langle N_{\tau^*}, \sigma_{\overline{\tau}\tau^*} \mid \overline{\tau} \leq \tau^*, \text{ and } \overline{\tau}, \tau^* \in D_\tau \cap \tilde{\tau} \rangle$. The direct limit $\tilde{N}$ is well-founded as we can define a map σ from $\tilde{N}$ into N_τ in an obvious manner. So from now on, we identify $\tilde{N}$ with its transitive collapse.

The following properties are evident: for $\tau^* \in D_\tau \cap \tilde{\tau}$,

- $\sigma_{\tau^*\tilde{\tau}}$ is $\Sigma_0^{(n)}$-preserving;
- $\sigma \circ \sigma_{\tau^*\tilde{\tau}} = \sigma_{\tau^*\tau}$;
- $\tilde{\tau} = \sigma_{\tau^*\tilde{\tau}}(\tau^*)$;
- $\tilde{\tau} = cr(\sigma)$ and $\sigma(\tilde{\tau}) = \tau$.

To see that $\tilde{N}$ is a premouse of the same type as N_τ, just notice that the direct limit maps (into $\tilde{N}$) preserve Π_2 properties upwards on a tail-end (cf. [3, Lemma 3.8]); $\tilde{N}$ is also Θ-g-organized by [4, Lemma 3.10] since N_τ is such and σ is sufficiently elementary.

Now let $\tilde{p} = \sigma_{\tau^*\tilde{\tau}}(p_{\tau^*})$ for some $\tau^* < \tilde{\tau}$. Given $x \in \tilde{N}$, there is an $x^* \in N_{\tau^*}$ for some $\tau^* \in D_\tau \cap \tilde{\tau}$ such that $\sigma_{\tau^*\tilde{\tau}}(x^*) = x$. By $\mathbb{R}$-soundness of N_{τ^*}, $x^* = \tilde{h}^{n+1}_{\tau^*}(a, p_{\tau^*})$ for some $a \in \mathbb{R}$. Since $\Sigma_1^{(n)}$ facts are preserved upwards by the direct limit maps, $x = \tilde{h}^{n+1}_{\tilde{N}}(a, \tilde{p})$. This implies $\tilde{N} = \tilde{h}^{n+1}_{\tilde{N}}(\mathbb{R} \cup \tilde{p})$, hence $\omega\rho^{n+1}_{\tilde{N}} = \mathbb{R}$ and $\tilde{p} \in R^{n+1}_{\tilde{N}}$, that is, $\tilde{p}$ is a very good parameter for $\tilde{N}$.

Next, we need to see that $\tilde{p}$ is the standard parameter for $\tilde{N}$ to conclude that $\tilde{N}$ is $\mathbb{R}$-sound. To see this, notice that $rng(\sigma_{\tau^*\tau}) \subset rng(\sigma)$, $\sigma(\tilde{p}) = p_\tau$, and $\sigma_{\tau^*\tau}(p_{\tau^*}) = p_\tau$ for any $\tau^* \in D_\tau \cap \tilde{\tau}$. This implies that $rng(\sigma)$ contains a generalized witness for each $\alpha \in p_\tau$ with respect to N_τ and p_τ. The above facts imply that $\tilde{p}$ is indeed the standard parameter for $\tilde{N}$, and hence $\tilde{N}$ is $\mathbb{R}$-sound. Consequently, $\tilde{N} = N_{\tilde{\tau}}$ and $\sigma = \sigma_{\tilde{\tau}\tau}$ by a similar interpolation argument as in Lemma 2.6. This gives us that $\tilde{\tau} \in D_\tau$.

□

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