SUPERCOMPACTNESS CAN BE EQUICONSISTENT WITH MEASURABILITY *†

NAM TRANG
Department of Mathematics
University of California, Irvine
Irvine, CA, 92697
Email: ntrang@math.uci.edu

Abstract

The main result of this paper, built on work of [15] and [14], is the proof that the theory "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ + there is an $\mathbb{R}$-complete measure on Θ" is equiconsistent with "$\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}}$ + there is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ + Θ is regular."

1. INTRODUCTION

We begin with the following definitions.

Definition 1.1 (ZF+DC). *Suppose X is an uncountable set and μ is a measure on $\wp_{\omega_1}(X) =_{def} \{\sigma \subseteq X \mid \sigma$ is countable$\}$. We say that*

1. *μ is **fine** if whenever $x \in X$, then the set $A_x =_{def} \{\sigma \mid x \in \sigma\} \in \mu$.*
2. *μ is **countably complete** if whenever $\langle A_n \mid n < \omega\rangle$ is a sequence of μ-measure one sets then $\bigcap_n A_n \in \mu$.*
3. *μ is **normal** if whenever $F : \wp_{\omega_1}(X) \to \wp_{\omega_1}(X)$ is such that the set $\{\sigma \mid F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mu$ then there is an $x \in X$ such that the set $\{\sigma \mid x \in F(\sigma)\} \in \mu$.*

If there is a nonprincipal measure μ on $\wp_{\omega_1}(X)$ that satisfies (1)-(3), then we say that ω_1 is X-supercompact. If there is a nonprincipal measure μ on $\wp_{\omega_1}(X)$ that satisfies (1) and (2) then we say ω_1 is X-strongly compact.

This is a generalization of the notion of supercompactness in the ZFC context. If X is a set of ordinals then the two notions coincide. The following is not hard to prove (see [13]).

Lemma 1.2 (ZF + DC). *Suppose μ is a fine measure on $\wp_{\omega_1}(X)$. The following are equivalent.*

1. *μ is normal.*

*2000 Mathematics Subject Classifications: 03E15, 03E45, 03E60.
†Keywords: Mouse, inner model theory, descriptive set theory, hod mouse, supercompactness on ω_1.

2. *Suppose we have $\langle A_x \mid x \in X \wedge A_x \in \mu \rangle$. Then $\triangle_{x\in X} A_x =_{def} \{\sigma \mid \sigma \in \bigcap_{x\in\sigma} A_x\} \in \mu$.*

From now on, the phrase "μ is a supercompact measure on $\wp_{\omega_1}(X)$" always means "μ is a nonprincipal, normal fine, countably complete measure on $\wp_{\omega_1}(X)$". We will also say "ω_1 is X-supercompact" to mean "there is a supercompact measure on $\wp_{\omega_1}(X)$". When μ is nonprincipal, countably complete, and fine (but not necessarily normal), we say that μ is a strongly compact measure. We say that ω_1 *is supercompact* if ω_1 is X-supercompact for all uncountable X and ω_1 *is strongly compact* if ω_1 is X-strongly compact for all uncountable X.

This paper explores aspects of compactness properties of ω_1 under $\mathsf{ZF}+\mathsf{DC}$. In particular, we focus on the consistency strength of the theories:

$$(\text{P}) \equiv \text{“}\mathsf{ZF} + \mathsf{DC} + \omega_1 \text{ is supercompact”},$$

$$(\text{Q}) \equiv \text{“}\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}_{\mathbb{R}} + \omega_1 \text{ is supercompact”}$$

and their variations. From here on, by $\mathsf{AD}_{\mathbb{R}}$, we always mean $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$. See Section 2 for basic terminology and facts about AD^+.

We note that "$\mathsf{ZF}+\omega_1$ is supercompact" implies DC by an easy argument. We choose to be redundant here since we'll be using DC in many arguments to come. Also, (Q) is equivalent to "$\mathsf{AD}^+ + \omega_1$ is supercompact" by results in [15] and [17].

Woodin (unpublished) has shown that Con(P) and Con(Q) follows from Con(ZFC + there is a proper class of Woodin limits of Woodin cardinals). We conjecture that a (closed to optimal) lower-bound consistency strength for the theory (P) is that of (Q) and is "ZFC + there is a Woodin limit of Woodin cardinals."

In the context of $\mathsf{ZF}+\mathsf{DC}$, the papers [14] and [12] study supercompact measures on $\wp_{\omega_1}(\mathbb{R})$ and show that the following theories are equiconsistent:

1. ZFC + there are ω^2 Woodin cardinals.
2. $\mathsf{AD}^+ +$ there is a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$.
3. $\mathsf{ZF} + \mathsf{DC} + \Theta > \omega_2 +$ there is a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$.[1]

It is also well-known that the existence of a supercompact measure on $\wp_{\omega_1}(\mathbb{R})$ is equiconsistent with that of a measurable cardinal (see [12]). Recall that the existence of supercompact measures on $\wp_{\omega_1}(\mathbb{R})$ was first shown by Solovay [8] from $\mathsf{AD}_{\mathbb{R}}$. Consistency-wise, it is known that $\mathsf{AD}_{\mathbb{R}}$ is much stronger than (1) (and hence (2) and (3)).

Surprisingly, [15] shows that having a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ is much stronger consistency-wise as it implies that there are models of $\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$. Solovay [8] shows that $\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$ is strictly stronger than $\mathsf{AD}_{\mathbb{R}}$ consistency-wise.

[1] The equiconsistency of (1) and (2) is due to H.W. Woodin. The equiconsistency of (2) and (3) is due independently to H.W. Woodin and the author.

Theorem 1.3 (Trang-Wilson)**.** *Assume* $\mathsf{ZF}+\mathsf{DC}$*. Suppose there is a supercompact measure on* $\wp_{\omega_1}(\wp(\mathbb{R}))$*. Then there is a transitive* M *containing* $\mathbb{R}\cup \mathrm{OR}\subseteq M$ *such that* $M\vDash \mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$*.*

[15] also shows the conclusion of Theorem 1.3 is equiconsistent with "$\mathsf{ZF}+\mathsf{DC}+\omega_1$ is $\wp(\mathbb{R})$-strongly compact". The main conjecture regarding compactness properties of ω_1 under $\mathsf{ZF}+\mathsf{DC}$ is.

Conjecture 1.4. *The following theories are equiconsistent.*

1. (P)
2. *"*$\mathsf{ZF}+\mathsf{DC}+\omega_1$ *is strongly compact"*

Conjecture 1.4's analogue in the ZFC context is perhaps more well-known. However, recent progress in inner model theory suggests that Conjecture 1.4 is more tractable.

Definition 1.5 (ZF+DC)**.** *Let* $\Theta=\sup(\{\alpha\mid \exists\pi:\mathbb{R}\to\alpha\wedge\pi \textit{ is onto}\})$ *and* μ *be a measure on* Θ*. We say that* μ *is* ***uniform*** *if sets of the form* $(\alpha,\Theta),[\alpha,\Theta)$ *are in* μ *for all* $\alpha<\Theta$*. We say that* μ *is* $\mathbb{R}$***-complete*** *if* μ *is uniform, and whenever we have* $\langle A_x\mid x\in\mathbb{R}\wedge A_x\in\mu\rangle$ *then* $\bigcap_{x\in\mathbb{R}}A_x\neq\emptyset$*.*

Let

- $(\mathrm{T}_1)\equiv$ "ZF + DC + there is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ + Θ is regular."
- $(\mathrm{T}_2)\equiv$ "$\mathsf{AD}_{\mathbb{R}}$ + DC + there is a nonprincipal $\mathbb{R}$-complete measure on Θ".
- $(\mathrm{T}_3)\equiv$ "$\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}_{\mathbb{R}}$ + there is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$ + Θ is regular."

We will also say "Θ is measurable" in place of "there is a nonprincipal $\mathbb{R}$-complete measure on Θ." The main theorem of this paper is the following.

Theorem 1.6. *Con(*T_2*)* $\Leftrightarrow$ *Con(*T_3*).*

The proof that (T_2) implies (T_3) (and hence (T_1)) is given in [13] (note that by a standard argument, Θ is measurable implies Θ is regular).[2] Hence we focus on the proof of Con(T_3) implies Con(T_2) in this paper.

Recent developments in the core model induction techniques suggest that the use of AD^+ in the proof of Theorem 1.6 can be omitted. We conjecture the following.

Conjecture 1.7. *Con(*T_1*)* $\Leftrightarrow$ *Con(*T_2*)(*$\Leftrightarrow$ *Con(*T_3*)). Furthermore, Con(P) implies Con(*T_3*).*

The outline of the paper is as follows. In Section 2, we summarize some basic facts about descriptive set theory and the theory of AD^+ that we use in this paper. Section 3 introduces the notion of hod mice that we will construct in this paper. Section 4 discusses a variation of the

[2] Let μ witness Θ is measurable. Suppose Θ is singular. Then it is easy to see that there is a cofinal map $f:\mathbb{R}\to\Theta$. For each $x\in\mathbb{R}$, let $A_x=\langle\alpha<\Theta\mid\alpha\geq f(x)\rangle$. Clearly $A_x\in\mu$ for all $x\in\mathbb{R}$. Let $\alpha\in\bigcap_x A_x\neq\emptyset$. Then $\alpha\geq f(x)$ for all $x\in\mathbb{R}$. This contradicts the fact that f is cofinal.

Vopenka algebra that is useful in constructing models of determinacy from hod mice (see Theorem 4.1). Section 5 gives the construction of a proper hod pair, which in turns will generate a model of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" and hence completes the proof of Theorem 1.6.

Acknowledgement. Part of this paper is from the author's PhD thesis; the author would like to thank G. Sargsyan and J.R. Steel for many helpful conversations regarding this topic while the author was a student at UC Berkeley.

2. BASIC FACTS ABOUT AD^+

We start with the definition of Woodin's theory of AD^+. In this paper, we identify $\mathbb{R}$ with ω^ω. Recall Θ is the sup of ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$. Under AC, Θ is just the successor cardinal of the continuum. In the context of AD, Θ is shown to be the supremum of $w(A)$ for $A \subseteq \mathbb{R}$.[3] The definition of Θ relativizes to any determined pointclass[4] (with sufficient closure properties). For a pointclass Γ, we denote Θ for the sup of α such that there is a surjection from $\mathbb{R}$ onto α coded by a set of reals in Γ.

Recall that AD_X is determinacy for games in which player I and II take turns to play elements of X for ω-many rounds. If $X = \omega$, then $\mathsf{AD}_X = \mathsf{AD}$.

Definition 2.1. AD^+ *is the theory* $\mathsf{ZF} + \mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ *and*

1. *for every set of reals A, there are a set of ordinals S and a formula φ such that $x \in A \Leftrightarrow L[S,x] \vDash \varphi[S,x]$. (S,φ) is called an ∞-Borel code for A;*
2. *for every $\lambda < \Theta$, for every continuous $\pi : \lambda^\omega \to \omega^\omega$, for every $A \subseteq \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.*

AD^+ is equivalent to "AD + the set of Suslin cardinals is closed". Another, perhaps more useful, characterization of AD^+ is "$\mathsf{AD}+\Sigma_1$ statements reflect into Suslin co-Suslin sets" (see [11] for the precise statement). Recall, our convention is $\mathsf{AD}_{\mathbb{R}}$ is the principle $\mathsf{AD}^+ + \mathsf{AD}_{\mathbb{R}}$.

Let $A \subseteq \mathbb{R}$, we let θ_A be the supremum of all α such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto α. If Γ is a determined (boldface) pointclass, and $A \in \Gamma$, we write $\Gamma \restriction A$ for the set of $B \in \Gamma$ which is Wadge reducible to A. If $\alpha < \Theta$, we write $\Gamma \restriction \alpha$ for the set of $A \in \Gamma$ with Wadge rank strictly less than α. Occasionally, we will write Γ for a ω-parameterized (lightface) pointclass and write $\utilde{\Gamma}$ for its corresponding boldface pointclass. We write $\utilde{\Delta}_{\utilde{\Omega}}$ for the ambiguous part of the boldface pointclass $\utilde{\Omega}$, that is $\utilde{\Delta}_{\utilde{\Omega}}$ is the collection of A such that both A and $\mathbb{R}\backslash A$ are in $\utilde{\Omega}$.

Definition 2.2 (AD^+)**.** *The* ***Solovay sequence*** *is the sequence $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ where*

1. *θ_0 is the supremum of ordinals β such that there is an OD surjection from $\mathbb{R}$ onto β;*
2. *$\theta_\Omega = \Theta$;*

[3] $w(A)$ is the Wadge rank of A.

[4] See [17] for more backgrounds on descriptive set theory in contexts where determinacy only holds locally.

3. *if $\alpha > 0$ is limit, then $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$;*

4. *if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e. $\beta < \Omega$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank θ_β, θ_α is the sup of ordinals γ such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto γ, i.e. $\theta_\alpha = \theta_A$.*

Note that the definition of θ_α for $\alpha = \beta + 1$ in Definition 2.2 does not depend on the choice of A. The Solovay sequence is a club set in Θ. Roughly speaking the longer the Solovay sequence is, the stronger the associated AD^+-theory is. For instance the theory $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$ is strictly stronger than $\mathsf{AD}_\mathbb{R}$ since by [8], DC implies $\mathrm{cof}(\Theta) > \omega$ while the minimal model of $\mathsf{AD}_\mathbb{R}$ satisfies $\Theta = \theta_\omega$ ($\mathsf{AD}_\mathbb{R}$ implies that the Solovay sequence has limit length). $\mathsf{AD}_\mathbb{R} + \Theta$ is regular is stronger still as it implies the existence of many models of $\mathsf{AD}_\mathbb{R} + \mathsf{DC}$.

Definition 2.3. *"$\mathsf{AD}_\mathbb{R} + \Theta$ is measurable" is the theory "$\mathsf{AD}_\mathbb{R}+$ there is a nonprincipal $\mathbb{R}$-complete measure on Θ".*

It's easy to see that "$\mathsf{AD}_\mathbb{R} + \Theta$ is measurable" implies "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular"; in fact, there are unboundedly many $\theta_\alpha < \Theta$ such that $L(\wp(\mathbb{R}) \restriction \theta_\alpha, \mathbb{R}) \vDash$ "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular".

We end this section with a theorem of Woodin, which produces models with Woodin cardinals in AD^+.

Theorem 2.4 (Woodin, see [4])**.** *Assume AD^+. Let $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ be the Solovay sequence. Suppose $\alpha = 0$ or $\alpha = \beta + 1$ for some $\beta < \Omega$. Then $\mathrm{HOD} \vDash \theta_\alpha$ is Woodin.*

3. A BRIEF INTRODUCTION TO HOD MICE

In this paper, *a hod premouse* $\mathcal{P}$ (below $\mathsf{AD}_\mathbb{R} + \Theta$ is measurable) is one defined as in [5]. The reader is advised to consult [5] for basic results and notations concerning hod premice and hod mice. Let us mention some basic first-order properties of a hod premouse $\mathcal{P}$. There are an ordinal $\lambda^\mathcal{P}$ and sequences $\langle (\mathcal{P}(\alpha), \Sigma^\mathcal{P}_\alpha) \mid \alpha < \lambda^\mathcal{P} \rangle$ and $\langle \delta^\mathcal{P}_\alpha \mid \alpha \leq \lambda^\mathcal{P} \rangle$ such that

1. $\langle \delta^\mathcal{P}_\alpha \mid \alpha \leq \lambda^\mathcal{P} \rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta^\mathcal{P}_\alpha$ is Woodin;

2. $\mathcal{P}(0) = Lp_\omega(\mathcal{P}|\delta_0)^\mathcal{P}$; for $\alpha < \lambda^\mathcal{P}$, $\mathcal{P}(\alpha+1) = (Lp_\omega^{\Sigma^\mathcal{P}_\alpha}(\mathcal{P}|\delta_\alpha))^\mathcal{P}$; for limit $\alpha \leq \lambda^\mathcal{P}$, $\mathcal{P}(\alpha) = (Lp_\omega^{\oplus_{\beta<\alpha}\Sigma^\mathcal{P}_\beta}(\mathcal{P}|\delta_\alpha))^\mathcal{P}$;

3. $\mathcal{P} \vDash \Sigma^\mathcal{P}_\alpha$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$[5]-strategy for $\mathcal{P}(\alpha)$ with hull condensation;

4. if $\alpha < \beta < \lambda^\mathcal{P}$ then $\Sigma^\mathcal{P}_\beta$ extends $\Sigma^\mathcal{P}_\alpha$.

We will write $\delta^\mathcal{P}$ for $\delta^\mathcal{P}_{\lambda^\mathcal{P}}$ and $\Sigma^\mathcal{P} = \oplus_{\beta<\lambda^\mathcal{P}} \Sigma^\mathcal{P}_\beta$. Note that $\mathcal{P}(0)$ is a pure extender model. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are two hod premice. Then $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ if there is $\alpha \leq \lambda^\mathcal{Q}$ such that $\mathcal{P} = \mathcal{Q}(\alpha)$. We say then that $\mathcal{P}$ is a *hod initial segment* of $\mathcal{Q}$. $(\mathcal{P}, \Sigma)$ is a *hod pair* if $\mathcal{P}$ is a hod premouse and Σ is a

[5]This just means $\Sigma^\mathcal{P}_\alpha$ acts on all stacks of ω-maximal, normal trees in $\mathcal{P}$.

strategy for $\mathcal{P}$ (acting on countable stacks of countable normal trees) such that $\Sigma^{\mathcal{P}} \subseteq \Sigma$ and this fact is preserved under Σ-iterations. Typically, we will construct hod pairs $(\mathcal{P}, \Sigma)$ such that Σ has hull condensation, branch condensation, and is Γ-fullness preserving for some pointclass Γ. As a matter of notation, if $(\mathcal{P}, \Sigma)$ is a hod pair and $\mathcal{Q} \lhd_{hod} \mathcal{P}$, then $\Sigma_{\mathcal{Q}}$ is Σ restricted to stacks on $\mathcal{Q}$.

Suppose $(\mathcal{Q}, \Sigma)$ is a hod pair such that Σ has hull condensation. $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse if there are ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle(\mathcal{P}(\alpha), \Sigma^{\mathcal{P}}_{\alpha}) \mid \alpha < \lambda^{\mathcal{P}}\rangle$ and $\langle \delta^{\mathcal{P}}_{\alpha} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ such that

1. $\langle \delta^{\mathcal{P}}_{\alpha} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta^{\mathcal{P}}_{\alpha}$ is Woodin;

2. $\mathcal{P}(0) = Lp^{\Sigma}_{\omega}(\mathcal{P}|\delta_0)^{\mathcal{P}}$ (so $\mathcal{P}(0)$ is a Σ-premouse built over $\mathcal{Q}$); for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (Lp^{\Sigma\oplus\Sigma^{\mathcal{P}}_{\alpha}}_{\omega}(\mathcal{P}|\delta_{\alpha}))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (Lp^{\Sigma\oplus_{\beta<\alpha}\Sigma^{\mathcal{P}}_{\beta}}_{\omega}(\mathcal{P}|\delta_{\alpha}))^{\mathcal{P}}$;

3. $\mathcal{P} \vDash \Sigma \cap \mathcal{P}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$strategy for $\mathcal{Q}$ with hull condensation;

4. $\mathcal{P} \vDash \Sigma^{\mathcal{P}}_{\alpha}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$strategy for $\mathcal{P}(\alpha)$ with hull condensation;

5. if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma^{\mathcal{P}}_{\beta}$ extends $\Sigma^{\mathcal{P}}_{\alpha}$.

Inside $\mathcal{P}$, the strategies $\Sigma^{\mathcal{P}}_{\alpha}$ act on stacks above $\mathcal{Q}$ and every Σ^{P}_{α} iterate is a Σ-premouse. Again, we write $\delta^{\mathcal{P}}$ for $\delta^{\mathcal{P}}_{\lambda^{\mathcal{P}}}$ and $\Sigma^{\mathcal{P}} = \oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma^{\mathcal{P}}_{\beta}$. $(\mathcal{P}, \Lambda)$ is a $(\mathcal{Q}, \Sigma)$-hod pair if $\mathcal{P}$ is a $(\mathcal{Q}, \Sigma)$-hod premouse and Λ is a strategy for $\mathcal{P}$ such that $\Sigma^{P} \subseteq \Lambda$ and this fact is preserved under Λ-iterations. The reader should consult [5] for the definition of $B(\mathcal{Q}, \Sigma)$, and $I(\mathcal{Q}, \Sigma)$. Roughly speaking, $B(\mathcal{Q}, \Sigma)$ is the collection of all hod pairs which are strict hod initial segments of a Σ-iterate of $\mathcal{Q}$ and $I(\mathcal{Q}, \Sigma)$ is the collection of all Σ-iterates of Σ. In the case $\lambda^{\mathcal{Q}}$ is limit, $\Gamma(\mathcal{Q}, \Sigma)$ is the collection of $A \subseteq \mathbb{R}$ such that A is Wadge reducible to some Ψ for which there is some $\mathcal{R}$ such that $(\mathcal{R}, \Psi) \in B(\mathcal{Q}, \Sigma)$. See [5] for the definition of $\Gamma(\mathcal{Q}, \Sigma)$ in the case $\lambda^{\mathcal{Q}}$ is a successor ordinal.

[5] constructs under AD^+ and the hypothesis that there are no models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" hod pairs that are fullness preserving, positional, commuting, and have branch condensation. Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD of AD^+ models. For hod pairs $(\mathcal{M}_{\Sigma}, \Sigma)$, if Σ is a strategy with branch condensation and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}_{\Sigma}$ with last model $\mathcal{N}$ (we will denote this model $\mathcal{N}^{\mathcal{T}}$), $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$ (this property is called *positional*). Therefore, later on we will omit the subscript $\vec{\mathcal{T}}$ from $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ whenever Σ is a strategy with branch condensation and $\mathcal{M}_{\Sigma}$ is a hod mouse. We also let $\alpha(\vec{\mathcal{T}})$ denote the supremum of the generators used in $\vec{\mathcal{T}}$.

Suppose AD^+ holds. We fix a simple (Δ^1_1) coding of H_{ω_1} by elements of $\mathbb{R}$. For an (ω_1, ω_1) iteration strategy Λ, we let Code(Λ) be the set of reals coding Λ via the specified coding.[6] Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is Γ-fullness preserving for some pointclass Γ and suppose Code(Σ) is Suslin co-Suslin, then [5, Corollary 2.44] shows that Σ is positional and commuting. We can then compute the direct limit $\mathcal{M}_{\infty}(\mathcal{P}, \Sigma)$ of all Σ-iterates of $\mathcal{P}$.

[6]To save space, we will generally not make distinction between Λ and Code(Λ) in this paper.

In practice (in determinacy models where the HOD analysis can be carried out or in core model induction contexts) we construct hod pairs $(\mathcal{P},\Sigma)$ such that Σ has branch condensation and is Γ-fullness preserving for some pointclass Γ (if $\Gamma=\wp(\mathbb{R})$ then we simply say "fullness preserving"). We don't quite have at the moment $(\mathcal{P},\Sigma)$ is constructed an AD^+-model M such that $(\mathcal{Q},\Sigma)\in M$[7] but we do know that every $(\mathcal{R},\Lambda)\in B(\mathcal{Q},\Sigma)$ belongs to such a model. We then can show (using our hypothesis) that the hod pair $(\mathcal{P},\Sigma)$ we construct belongs to an AD^+-model.

In this paper, $\mathcal{P}$ is a hod premouse if

(i) either $\mathcal{P}$ is a hod premouse below "$\mathsf{AD}_{\mathbb{R}}+\Theta$ is measurable", that is, no hod initial segment $\mathcal{Q}$ of $\mathcal{P}$ satisfies "$\delta^{\mathcal{Q}}$ is a measurable limit of Woodin cardinals" ($\mathcal{P}$ is called *improper* in this case),

(ii) or $\mathcal{P}=(\mathcal{P}^-,E)$ where $\mathcal{P}^-$ is improper hod premouse (or anomalous hod premouse, cf. [5, Section 3.4]), $\mathcal{P}\vDash$ "$\delta^{\mathcal{P}}$ is regular" and E codes (as an amenable predicate) a normal measure over $\mathcal{P}$ with critical point $\delta^{\mathcal{P}}$ ($\mathcal{P}$ is called *proper* in this case).

Suppose $\mathcal{P}$ is a proper hod premouse and suppose Σ is some iteration strategy of $\mathcal{P}$. Suppose $\vec{\mathcal{T}}$ is a stack according to Σ. It's easy to see that $\vec{\mathcal{T}}$ can be decomposed into a sequence of stacks $(\mathcal{T}_\alpha,\mathcal{N}_\alpha:\alpha<\gamma)$ for some γ, where

1. $\mathcal{N}_0=\mathcal{P}=(\mathcal{N}_0^-,E_0)$, $\mathcal{N}_{\alpha+1}$ is the last model of $\mathcal{T}_\alpha$, and for limit α, $\mathcal{N}_\alpha$ is the direct limit (under the iteration maps) of the $\mathcal{N}_\beta$'s for $\beta<\alpha$;
2. for $\alpha<\gamma-1$ successor, say $\mathcal{N}_\alpha=(\mathcal{N}_\alpha^-,E_\alpha)$. Then $\mathcal{T}_{\alpha+1}$ is either a stack below $\delta^{\mathcal{N}_\alpha}$ (if $\mathcal{T}_\alpha=\langle\mathcal{N}_{\alpha-1}^-,E_{\alpha-1}\rangle$) or else $\mathcal{T}_{\alpha+1}=\langle\mathcal{N}_\alpha^-,E_\alpha\rangle$.
3. for $\alpha=0$ or limit, $\mathcal{T}_\alpha$ is either a stack on $\mathcal{N}_\alpha$ below $\mathcal{N}_\alpha$ or else $\mathcal{T}_\alpha=\langle\mathcal{N}_\alpha^-,E_\alpha\rangle$;

Such a sequence is called *the normal form* of $\vec{\mathcal{T}}$.

4. A VOPENKA FORCING

In this section, we prove a theorem concerning a variation of the Vopenka algebra. This theorem will play an important role in the next section. Suppose Γ is such that $L(\Gamma,\mathbb{R})\vDash\mathsf{AD}^++\mathsf{AD}_{\mathbb{R}}$ and $\Gamma=\wp(\mathbb{R})\cap L(\Gamma,\mathbb{R})$. Let $\mathcal{H}$ be $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. Woodin has shown that $\mathcal{H}=L[A]$ for some $A\subseteq\Theta$ (see [16]). We write Θ for Θ. Let $\mathcal{H}^+$ be a ZFC model such that $A\in\mathcal{H}^+$ and $V_\Theta^{\mathcal{H}}=V_\Theta^{\mathcal{H}^+}$. The following theorem comes from many conversations between H.W. Woodin and the author.

Theorem 4.1. *There is a forcing $\mathbb{P}\in\mathcal{H}$ such that*

1. *$\mathbb{P}$ is homogeneous;*
2. *there is a $G\subseteq\mathbb{P}$ generic over $\mathcal{H}^+$ such that $\mathcal{H}^+(\Gamma)$ is the symmetric part of $\mathcal{H}^+[G]$.*[8]

[7] Though in the proof of Theorem 1.6 we do know this.

[8] $\mathcal{H}^+(\Gamma)$ is the minimal transitive ZF model containing $\mathcal{H}^+$ and Γ. The symmetric part of $\mathcal{H}[G]$ is the set of τ_G where τ is a $\mathbb{P}$-symmetric name. Another way of defining the symmetric part of $\mathcal{H}[G]$ is $\bigcup_{\alpha<\Theta}\mathrm{HOD}^{\{\mathcal{H}[G],\mathcal{H}\}}_{\{G\cap V_\alpha^{\mathcal{H}}\}}$.

3. $\wp(\mathbb{R}) \cap \mathcal{H}^+(\Gamma) = \Gamma$.

In particular, $\mathcal{H}^+(\Gamma) \models \mathsf{AD}_{\mathbb{R}}$.

Proof. First, we define a forcing $\mathbb{Q} \in L(\Gamma, \mathbb{R})$. A condition $q \in \mathbb{Q}$ if $q : n_q \to \wp(\alpha_q)$ for some $n_q < \omega$ and $\alpha_q < \Theta$. The ordering $\leq_{\mathbb{Q}}$ is as follows:

$$q \leq_{\mathbb{Q}} r \Leftrightarrow n_r \leq n_q \wedge \alpha_r \leq \alpha_q \wedge \forall i < n_r \; q(i) \cap \alpha_r = r(i).$$

Now we define

$$\mathbb{P}^* = \{A \mid \exists \alpha_A < \Theta \exists n_A < \omega \; A \subsetneq \wp(\alpha_A)^{n_A} \wedge A \in OD^{L(\Gamma,\mathbb{R})}\}.$$

The ordering $\leq_{\mathbb{P}^*}$ is defined as follows:

$$A \leq_{\mathbb{P}^*} B \Leftrightarrow n_B \leq n_A \wedge \alpha_B \leq \alpha_A \wedge A \cap \wp(\alpha_B)^{n_B} \subseteq B.$$

In the above, by "$A \cap \wp(\alpha_B)^{n_B}$", we mean the set $\{x \restriction n_B \mid \exists y \in A \; x = \langle y(0) \cap \alpha_B, \dots, y(n_A - 1) \cap \alpha_B \rangle\}$.

It's easy to see that there is a partial order $(\mathbb{P}, \leq_{\mathbb{P}}) \in \mathcal{H}$ isomorphic to $(\mathbb{P}^*, \leq_{\mathbb{P}^*})$ and in $\mathcal{H}$, $(\mathbb{P}, \leq_{\mathbb{P}})$ has size Θ. Let $\pi : (\mathbb{P}, \leq_{\mathbb{P}}) \to (\mathbb{P}^*, \leq_{\mathbb{P}^*})$ be the isomorphism (and π is $OD^{L(\Gamma,\mathbb{R})}$). We will write p^* for $\pi(p)$, where $p \in \mathbb{P}$. We will occasionally confuse these two partial orders. $(\mathbb{P}, \leq_{\mathbb{P}})$ is the direct limit of the directed system of complete boolean algebras $\mathbb{P}_{\alpha,n}$ in $\mathcal{H}$, where $\mathbb{P}^*_{\alpha,n}$ is the Vopenka algebra on $\wp(\alpha)^n$ and the maps from $\mathbb{P}_{\alpha,n}$ into $\mathbb{P}_{\beta,m}$ for $\alpha \leq \beta$ and $n \leq m$ are the natural maps. It's clear that $\mathbb{P}$ is homogeneous. Similarly, $\mathbb{Q}$ is homogeneous and is a natural direct limit of the partial orders $\{\mathbb{Q}_\alpha = \mathbb{Q} \restriction \alpha \mid \alpha < \Theta\}$, where $\mathbb{Q} \restriction \alpha$ consists of conditions p in $\mathbb{Q}$ such that $\alpha_p \leq \alpha$.

Now let $g \subseteq \mathbb{Q}$ be $L(\Gamma, \mathbb{R})$-generic. Let $h \subseteq \mathbb{P}$ be defined as follows:

$$p \in h \Leftrightarrow (g \restriction n_{p^*}) \cap \wp(\alpha_{p^*})^{n_{p^*}} \in p^*.$$

We can think of g as a function from $\Theta \times \omega$ to $\wp_\Theta(\Theta)$ such that for all $(\alpha, n) \in \mathrm{dom}(g)$, $g(\alpha, n) \subseteq \alpha$. Also, if $p \in \mathbb{P}$, by (n_p, α_p), we mean (n_{p^*}, α_{p^*}).

Lemma 4.2. *Write the filter h above h_g. Then h_g is generic over $\mathcal{H}$ and $L(\Gamma, \mathbb{R})$ is the symmetric part of $\mathcal{H}[h_g]$. In fact, for any condition $p \in \mathbb{P}$, there is a generic filter h over $\mathcal{H}$ such that $p \in h$ and $L(\Gamma, \mathbb{R})$ is the symmetric part of $\mathcal{H}[h]$.*

Proof. Suppose h_g is not generic over $\mathcal{H}$. Then there is an open dense set $D \subseteq \mathbb{P}$ in $\mathcal{H}$ such that $h_g \cap D = \emptyset$. Fix a condition $p \in g$ which forces this. For each $i < \omega$, let p_i be the join in $\mathbb{P}_{\alpha_p,i}$ of $D_{p,i}$, where $D_{p,i} \subseteq \mathbb{P}_{\alpha_p,i}$ is the set of all b which can be refined in $\mathbb{P}$ to an element of D by not increasing i but (possibly) increasing α_p, i.e. there is a $\beta \geq \alpha_p$ and a $d \in \mathbb{P}_{\beta,i}$ such that $d \in D$ and $d \restriction \alpha_p = b$.

Since D is open dense, the set $\{p_i \mid i < \omega\}$ is predense in the limit $\mathbb{P}_{\alpha_p}$ of the $\mathbb{P}_{\alpha_p,i}$'s (this just says that $\bigcup_i D_{p,i}$ is predense in $\bigcup_i \mathbb{P}_{\alpha_p,i}$). Since $g \restriction \alpha_p =_{def} \langle g(n) \restriction \alpha_p \mid n < \omega \rangle$ is generic for $\mathbb{Q} \restriction \alpha_p$, there must be some $i \geq n_p$ and $\beta \geq \alpha_p$ such that there is some $b \in \mathbb{P}_{\beta,i} \cap D$ such that $(g \restriction \alpha_p) \restriction i \in b^* \restriction \wp(\alpha_p)^i$. But this means we can easily refine p to a condition q such that

$$q \Vdash \dot{h}_{\dot{g}} \cap D \neq \emptyset,$$

by taking q to be a "thread" in b^* extending $(g \restriction \alpha_p) \restriction i$. This is a contradiction.

In fact, we just proved that given an open dense set $D \subseteq \mathbb{P}$ in $\mathcal{H}$, for any condition $p \in \mathbb{Q}$, there is a $q \leq_{\mathbb{Q}} p$ such that $q \Vdash_{\mathbb{Q}} \dot{h} \cap D \neq \emptyset$. Given g and h_g as above, we also can define g from h_g in a simple way. Let $b \subseteq \alpha$ for some $\alpha < \Theta$ such that $b \in L(\Gamma, \mathbb{R})$. Let $p_{b,\alpha,n} \in \mathbb{P}$ be such that $n_{p_{b,\alpha,n}} = n+1$ (for some n) and $\alpha_{p_{b,\alpha,n}} = \alpha$ and $b \in (p_{b,\alpha,n})^*(n)$. We can pick a map $\langle b, \alpha, n\rangle \mapsto p_{b,\alpha,n}$ that is OD in $L(\Gamma, \mathbb{R})$. Then

$$b = g(\alpha, n) \Leftrightarrow p_{b,\alpha,n} \in h_g.$$

We then can define symmetric $\mathbb{P}$-terms for $h_g(\alpha, n)$ and $ran(h_g)$ by

$$\sigma_{\alpha,n} = \{\langle p, \check{b}\rangle \mid b \subseteq \alpha \wedge p \leq_{\mathbb{P}} p_{b,\alpha,n}\},$$

and

$$\dot{R} = \{\langle p, \sigma_{\alpha,n}\rangle \mid p \in \mathbb{P} \wedge \alpha < \Theta \wedge n < \omega\}.$$

By the proof above, we have the following.

Lemma 4.3. *1. For any $g \subseteq \mathbb{Q}$ generic over $L(\Gamma, \mathbb{R})$, $\sigma_{\alpha,n}^{h_g} = h_g(\alpha, n)$ for all α, n and $\dot{R}^{h_g} = ran(h) = (\wp_\Theta(\Theta))^{L(\Gamma,\mathbb{R})}$.*

2. For any condition $p \in \mathbb{P}$, there is an $\mathcal{H}$-generic h such that $p \in h$ and $\dot{R}^h = \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$.

Since $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}^+ + \mathsf{AD}_\mathbb{R}$, $L(\Gamma, \mathbb{R})$ can be recovered over $\mathcal{H}$ from $\wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$ (via the standard Vopenka forcing).[9] This and Lemma 4.3 prove Lemma 4.2. □

Now work in $L(\mathcal{H}^+, g)$ for a $\mathbb{P}$-generic g over $\mathcal{H}$ such that $\dot{R}^g = \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$. It makes sense then to talk about the forcing $\mathbb{Q}$ in the model $L(\mathcal{H}^+, g)$. Also, note that $\mathbb{P} \in \mathcal{H}^+$. The following lemma is the key lemma.

Lemma 4.4. *There is a $\mathbb{P}$-generic g^* over $\mathcal{H}^+$ such that*

1. *$\dot{R}^{g^*} = \dot{R}^g = \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$.*

2. *$\mathcal{H}^+(\dot{R}^{g^*}) \cap \wp_\Theta(\Theta) = \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$ and $\mathcal{H}^+(\dot{R}^{g^*}) \cap \wp(\mathbb{R}) = \Gamma$.*

Proof. Let $h^* \subseteq \mathbb{Q}$ be $L(\mathcal{H}^+, g)$-generic. As mentioned above, $\mathbb{Q} \in L(\mathcal{H}^+, g)$ since $\dot{R}^g = \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$. Now, let $g^* = g_{h^*}$. Using the proof of Lemma 4.2 and the fact that $V_\Theta^{\mathcal{H}^+} = V_\Theta^{\mathcal{H}}$, we get that g^* is generic over $\mathcal{H}^+$ and $\dot{R}^{g^*} = \dot{R}^g$.

Now we want to verify clause (2) of the lemma. For the first equality, it's clear that the $\supseteq$-direction holds. For the converse, if A is a bounded subset of Θ in $\mathcal{H}^+(\dot{R}^{g^*})$ (so A has a symmetric name in $\mathcal{H}^+$), then using the automorphisms of $\mathbb{P}$ that are in $\mathcal{H}$, it's easy to see that there is some $\alpha < \Theta$ such that $A \in \mathcal{H}^+[g^* \restriction \alpha]$. To see this, note that if $p_0, p_1 \in \mathbb{P}$ decide differently the statement

[9] Any $A \in \Gamma$ has an ∞-Borel code $S \in \wp_\Theta(\Theta)$. S is generic over $\mathcal{H}$ via a (Vopenka) forcing of size $< \Theta$ (this uses that $L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}_\mathbb{R}$). Then there is a formula φ such that given any real x, $\mathcal{H}[S][x] \vDash \varphi[S, x]$ if and only if $x \in A$.

"$\check{\beta} \in \dot{A}$", then by homogeneity, there is an automorphism in $\mathcal{H}$ that maps p_0 to a p_0' compatible with p_1. This is a contradiction. Now since $\mathbb{P} \restriction \alpha$ is Θ-c.c. (by $\mathsf{AD}_{\mathbb{R}}$)[10] and $V_\Theta^{\mathcal{H}^+} = V_\Theta^{\mathcal{H}}$, $A \in \mathcal{H}(\dot{R}^g)$, and hence $A \in \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$.

Note that the first equality of (2) shows that $\mathbb{R} \cap \mathcal{H}^+(\dot{R}^{g^*}) = \mathbb{R}^V$. Now we're onto the second equality of (2). The $\supseteq$-direction holds since $\mathcal{H}(\wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}) = L(\Gamma, \mathbb{R}) \subseteq \mathcal{H}^+(\dot{R}^{g^*})$. Let $A \subseteq \mathbb{R}^V$ be in $\mathcal{H}^+(\dot{R}^{g^*})$. First we assume A is definable in $\mathcal{H}^+(\dot{R}^{g^*})$ from an element $a \in \mathcal{H}^+$, via a formula ψ. Let $\dot{x}$ be a (symmetric) $\mathbb{P} \restriction \omega$-name for a real. The statement $\psi(\dot{x}, \check{a})$ is decided by $\mathbb{P} \restriction \omega$ by homogeneity of $\mathbb{P} \restriction \omega$ (i.e. $\mathcal{H}^+ \vDash$ "$\emptyset \Vdash_{\mathbb{P}\restriction\omega} \psi[\dot{x}, \check{a}] \vee \emptyset \Vdash_{\mathbb{P}\restriction\omega} \neg\psi[\dot{x}, \check{a}]$"). Again, by the fact that $\mathbb{P} \restriction \omega$ is Θ-c.c., we get that $A \in \mathcal{H}[g^* \restriction \omega]$, hence $A \in \Gamma$.

Now suppose A is definable in $\mathcal{H}^+(\dot{R}^{g^*})$ from an $a \in \mathcal{H}^+$ and a $b \in \wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})}$. Using the standard Vopenka algebra and $\mathsf{AD}_{\mathbb{R}}$, we can get a $<\Theta$-generic G_b over $\mathcal{H}$ and $\mathcal{H}^+$ such that $\mathrm{HOD}_b^{L(\Gamma,\mathbb{R})} = \mathcal{H}[G_b] \subseteq \mathcal{H}^+[G_b]$. Let us use $\mathcal{H}_b$ to denote $\mathcal{H}[G_b]$ and $\mathcal{H}_b^+$ to denote $\mathcal{H}^+[G_b]$. Now in $\mathcal{H}_b$, we can define the poset $\mathbb{P}_b$ the same way that $\mathbb{P}$ defined but we replace OD by $OD(b)$ in $L(\Gamma, \mathbb{R})$. Let g_b be $\mathbb{P}_b$-generic over $\mathcal{H}_b$ such that the symmetric part of $\mathcal{H}_b[g_b]$ is $\mathcal{H}_b(\wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})})$. Now we get a generic g_b^* over $\mathcal{H}_b^+$ from g_b as before. A is then definable over $\mathcal{H}_b^+(\wp_\Theta(\Theta)^{L(\Gamma,\mathbb{R})})$ from parameters in $\mathcal{H}_b^+$. Now, we just have to repeat the argument above. This completes the proof of Lemma 4.4. □

Lemmata 4.2, 4.3, and 4.4 together prove Theorem 4.1. □

We end this section with the following lemma, which will be useful in the next section.

Lemma 4.5. *Suppose $M = L(\Gamma, \mathbb{R}) \vDash \mathsf{AD}_{\mathbb{R}}$. Letting $\mathcal{H} = \mathrm{HOD}^{\mathrm{M}}$ and say, $\mathcal{H} = L[A]$ for some $A \subseteq \Theta^M$. Let $\mathcal{H}^+$ be a ZFC transitive model such that $V_{\Theta^M}^{\mathcal{H}^+} = V_{\Theta^M}^{\mathcal{H}}$ and $A \in \mathcal{H}^+$. Let $\mathbb{P}$ be the Vopenka forcing defined in Theorem 4.1. Suppose $\mathcal{H}^+ \vDash \Theta^M$ is regular. Then $L[\mathcal{H}^+](\Gamma) \vDash \Theta^M$ is regular.*

Proof. Let $G \subseteq \mathbb{P}$ be $\mathcal{H}^+$-generic such that the conclusions of Lemma 4.4 hold for G. Suppose $\alpha < \Theta^M$ and $f : \alpha \to \Theta^M$ is a cofinal function in $L[\mathcal{H}^+](\Gamma)$. Say f is definable via φ over $L[\mathcal{H}^+](\Gamma)$ from ordinals and $\sigma^G_{\alpha_0,n_0}, \ldots, \sigma^G_{\alpha_k,n_k}$. It is easy to see that every $x \in L[\mathcal{H}^+](\Gamma)$ has this form.

Let $p \in G$ be such that $n_p \geq \max\{n_0, \ldots, n_k\}$, $\alpha_p \geq \max\{\alpha_0, \ldots, \alpha_k\}$ and p forces φ defines a cofinal map from $\check{\alpha}$ into Θ^M from $\sigma^G_{\alpha_0,n_0}, \ldots, \sigma^G_{\alpha_k,n_k}$. Note that if $q \leq p$ forces "$\varphi(\check{\xi}) = \check{\beta}$", [11] then $q \cap \wp(\alpha_p)^{n_p}$ forces "$\varphi(\check{\xi}) = \check{\beta}$". One simply uses the automorphisms of $\mathbb{P}$ that permute coordinates to see this.

But by $\mathsf{AD}_{\mathbb{R}}$, there are less than Θ^M many possible $q \cap \wp(\alpha_p)^{n_p}$ in M. Since $\mathcal{H}^+ \vDash \Theta^M$ is regular, this easily gives a contradiction. □

[10] By $\mathsf{AD}_{\mathbb{R}}$, letting $\gamma = \theta_\beta$ for some $\theta_\beta > \alpha$, then $\mathbb{P} \restriction \alpha \in L(\Gamma \restriction \gamma, \mathbb{R})$ and every antichain of $\mathbb{P} \restriction \alpha$ is in $L(\Gamma \restriction \gamma, \mathbb{R})$. Θ-cc-ness of $\mathbb{P} \restriction \alpha$ easily follows.

[11] This just means the value of the function defined by φ at ξ is β. We write this to make the notation less cumbersome.

5. A PROOF OF THEOREM 1.6

In this section, we assume the hypothesis of Theorem 1.6. We start with some setup and notations. As in [15], we assume $V = L(\wp(\mathbb{R}), \mu)$, where "$\mathsf{AD}_{\mathbb{R}} + \mathsf{DC} + \Theta$ is regular" holds and μ is a supercompact measure on $\wp_{\omega_1}(\wp(\mathbb{R}))$. Suppose N is such that there is a surjection π^* from $\wp(\mathbb{R})$ onto N. Then π^* induced a surjection $\pi : \wp_{\omega_1}(\wp(\mathbb{R})) \to \wp_{\omega_1}(N)$, namely $\pi(\sigma) = \pi^*[\sigma]$. Let μ_N be the supercompact measure on $\wp_{\omega_1}(N)$ induced by μ, i.e.

$$A \in \mu_N \Leftrightarrow \pi^{-1}[A] \in \mu.$$

In general, μ_N depends on π but we suppress it from the notation; for our purposes, any supercompact measure on $\wp_{\omega_1}(N)$ suffices. We write $\forall^*_{\mu_N}\sigma$ for "for μ_N-a.e. σ".

For each $\sigma \in \wp_{\omega_1}(N)$, let

$$\mathscr{M}_\sigma = \mathrm{HOD}^{(V,\mu_N)}_{\sigma \cup \{\sigma\}},$$

and

$$\mathscr{H}_\sigma = \mathrm{HOD}^{(\mathscr{M}_\sigma,\mu_N)}_{\{\sigma\}}.$$

We summarize a few facts that are proved in [15].

(i) The ultraproducts $\mathscr{M} = \prod_\sigma \mathscr{M}_\sigma/\mu_N$ and $\mathscr{H} = \prod_\sigma \mathscr{H}_\sigma/\mu_N$ are well-founded and Los theorem holds for these ultraproducts (cf. [15, Lemma 3.1] or [14, Lemma 2.4]).

(ii) $N \subseteq \mathscr{M}$. If $\wp(\mathbb{R}) \subseteq N$, then $\wp(\mathbb{R}) \subset \mathscr{M}$ and hence $\wp_\Theta(\Theta) \subset \mathscr{M}$.

(iii) Suppose N is a transitive structure of a fragment of ZF, then $\forall^*_{\mu_N}\sigma\ \sigma \prec N$.

(iv) ω_1^V is measurable in both $\mathscr{M}_\sigma$ and $\mathscr{H}_\sigma$ for μ_N-a.e. σ.

(v) For any $A \in \mathscr{M}$ such that A is a bounded subset of $\Omega =_{\mathrm{def}} [\sigma \mapsto \omega_1^V]_{\mu_N}$ in $\mathscr{M}$, A is generic over $\mathscr{H}$ via a forcing of size $< \Omega$. Also $\Omega > \Theta$.

We assume, for contradiction that

(†) : there is no model M containing all reals and ordinals such that $M \vDash$ "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable".

Under this smallness assumption, the HOD analysis in V can be carried out as in [5] to conclude that $\mathrm{HOD}|\Theta$ is a union of hod premice and in fact is a direct limit of the directed system $\mathcal{F}$ of hod pairs $(\mathcal{P}, \Sigma)$ such that Σ is fullness preserving and has branch condensation. We then construct a hod premouse $\mathcal{H}^+$ extending $\mathrm{HOD}|\Theta$ and a normal measure ν on Θ over $\mathcal{H}^+$ and amenable to $\mathcal{H}^+$. So we have a proper hod premouse $(\mathcal{H}^+, \nu)$. Using the Vopenka forcing in the previous section, we then show that $V = L[\mathcal{H}^+][\nu](\wp(\mathbb{R})) \vDash \mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable. This contradicts (†). So (†) must be false; equivalently, there must be models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable" after all.

We define a model $\mathcal{H}^+$ extending $\mathcal{H} =_{\mathrm{def}} \mathrm{HOD}|\Theta$ as follows. Let N be a transitive structure of a large fragment of $\mathsf{ZF} + \mathsf{DC}$ such that $\wp(\mathbb{R}) \cup \mathcal{H} \subset N$ and such that there is a surjection

$\pi : \wp(\mathbb{R}) \to N$. We have that $\forall^*_{\mu_N}\sigma\ \sigma \prec N$. For each such σ, let N_σ be the transitive collapse of σ and π_σ be the uncollapse map. Let $(\Gamma_\sigma, \mathcal{H}_\sigma, \Theta_\sigma) = \pi_\sigma^{-1}(\wp(\mathbb{R}), \mathcal{H}, \Theta)$. Generally, if $x \in \sigma$, then let $x_\sigma = \pi_\sigma^{-1}(x)$.

We let $\mathcal{H}_\sigma = \mathrm{HOD}^{M_\sigma}$, where M_σ is the transitive collapse of σ. We then let

$$\mathcal{H}^+ = \Pi_\sigma \mathrm{Lp}^{\Sigma_\sigma^-}(\mathcal{H}_\sigma)/\mu_{\mathrm{N}},$$

$$\mathcal{H}^{+-} = \Pi_\sigma \mathrm{Lp}_\omega^{\Sigma_\sigma^-}(\mathcal{H}_\sigma)/\mu_{\mathrm{N}},$$

where $\Sigma_\sigma^- = \oplus_{\alpha<\lambda_\sigma}\Sigma_{\mathcal{H}_\sigma(\alpha)}$. We also let $\mathcal{H}_\sigma^+ = \mathrm{Lp}_\omega^{\Sigma_\sigma^-}(\mathcal{H}_\sigma)$. The lemma below justifies the notation of $\mathcal{H}^+$ etc. that does not refer to the structure N.

Lemma 5.1. *The definition of $\mathcal{H}^+$ is independent of the choice of N.*

Proof. Suppose N_0, N_1 have the properties of N described above. Let $\mathcal{H}_i^+$ be defined like $\mathcal{H}^+$ but relative to N_i $(i = 0, 1)$. Without loss of generality, we assume $N_0 \subseteq N_1$ and $\mu_1 =_{\mathrm{def}} \mu_{N_1}$ projects to $\mu_0 =_{\mathrm{def}} \mu_{N_0}$.

Our assumption implies that $\mathcal{H} \lhd \mathcal{H}_0^+ \unlhd \mathcal{H}_1^+$ and $\mathcal{H} = [\sigma \mapsto \mathcal{H}_\sigma]_{\mu_0} = [\sigma \mapsto \mathcal{H}_\sigma]_{\mu_1}$.

Let $\mathcal{M} \lhd \mathcal{H}_1^+$ be such that $\rho_\omega \leq \Theta$ and let $p^{\mathcal{M}}$ be the standard parameter of $\mathcal{M}$. Since $\mathcal{M}$ is Θ-sound, the Σ_ω-theory $\mathrm{Th}_\omega^{\mathcal{M}}(\Theta \cup \{p^{\mathcal{M}}\})$ codes $\mathcal{M}$ and is essentially a subset of Θ (so we will confuse $\mathcal{M}$ with $\mathrm{Th}_\omega^{\mathcal{M}}(\Theta \cup \{p^{\mathcal{M}}\})$). Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_{\mu_1}$; again, we identify $\mathcal{M}_\sigma$ with its theory as above. Notice that $\forall^*_{\mu_1}\sigma\ \mathcal{M}_\sigma \in \mathcal{H}_{\sigma\cap N_0}$. This easily implies $\mathcal{M} \lhd \mathcal{H}_0^+$.

We have shown $\mathcal{H}_0^+ = \mathcal{H}_1^+$.

□

Lemma 5.2. *No level $\mathcal{M}$ of $\mathcal{H}^+$ is such that $\rho_\omega(\mathcal{M}) < \Theta$.*

Proof. We start with the following.

Claim 5.3. *For μ_N-a.e. σ, for any $\beta < \lambda_\sigma =_{\mathrm{def}} \lambda^{\mathcal{H}_\sigma}$, $\Sigma_{\mathcal{H}_\sigma(\beta)}$ is fullness preserving and has branch condensation.*

Proof. Fix a σ and a $\beta < \lambda_\sigma$. By the HOD analysis in Γ_σ (which uses (†)), there is a hod pair $(\mathcal{P}, \Sigma)$ such that

- Σ is Γ_σ-fullness preserving and has branch condensation;
- $\mathcal{H}_\sigma(\beta)$ is an iterate of Σ.

Using π_σ, we get that $\pi_\sigma(\Sigma)$ is an (ω_1, ω_1) strategy for $\mathcal{P}$ that is fullness preserving and has branch condensation. Since $\Sigma = \pi_\sigma(\Sigma) \restriction \Gamma_\sigma$, $\Sigma^{\mathcal{H}_\sigma(\beta)}$ is the tail of $\pi_\sigma(\Sigma)$ and hence satisfies the conclusion of the claim.[12] □

[12]Note that by positionality of $\pi_\sigma(\Sigma)$, which follows from fullness preservation and branch condensation (cf. [5, Theorem 2.42], $\Sigma_{\mathcal{H}_\sigma(\beta)}$ does not depend on any specific iteration from $\mathcal{P}$ to $\mathcal{H}_\sigma(\beta)$.

We are now ready to finish the proof of the lemma. Fix a σ in the claim. Let $\mathcal{H}^*_\sigma$ be the least level of $\mathcal{H}^+_\sigma$ that projects across Θ_σ. By choosing an $N' \supset N$ that contains also $\mathcal{H}^* = [\sigma \mapsto \mathcal{H}^*_\sigma]_{\mu_N}$ and using Lemma 5.1, we may assume $\forall^*\mu_N\sigma\ \mathcal{H}^*_\sigma \in N_\sigma$. Note that $\forall^*_{\mu_N}\sigma\ \sigma_\sigma^{-1}(\mathcal{H}^*) = \mathcal{H}^*_\sigma$.

Let Σ_σ be the natural strategy of $\mathcal{H}^*_\sigma$ defined from π_σ (see [6, Section 11]). The important properties of Σ_σ are:

1. Σ_σ extends Σ^-_σ and Σ_σ is $\mathrm{OD}_{\{\pi_\sigma \restriction \mathcal{H}^*_\sigma\}}$;

2. whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{H}^*_\sigma, \Sigma_\sigma)$, for all $\alpha < \lambda^{\mathcal{Q}}$, $\Sigma_{\mathcal{T},\mathcal{Q}(\alpha)}$ is the pullback of a hod pair $(\mathcal{R}, \Lambda)$ such that Λ has branch condensation and is fullness preserving and hence by [5, Lemma 3.29], $\Sigma_{\mathcal{T},\mathcal{Q}(\alpha)}$ has branch condensation;

3. Σ_σ agrees with Σ^-_σ on stacks below Θ_σ and for each $\alpha < \lambda_\sigma$, the direct limit map $\pi^{\Sigma_\sigma}_{\mathcal{H}^+_\sigma,\infty} \restriction (\theta_\sigma)_\alpha$ is the direct limit map $\pi^{\Sigma^-_\sigma}_{\mathcal{H}_\sigma(\alpha),\infty} \restriction (\theta_\sigma)_\alpha$;

4. suppose $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{H}^*_\sigma, \Sigma_\sigma)$ and let $i = \pi^{\vec{\mathcal{T}}}$ be the corresponding iteration map, then there is a map $k : \mathcal{Q} \to \mathcal{H}^*$ such that $k \circ i = \pi_\sigma \restriction \mathcal{H}^*_\sigma$. k is defined as: $k(i(f)(a)) = \pi_\sigma(f)(\pi^\Lambda_{\mathcal{Q},\infty}(a))$ for $f \in \mathcal{H}^*_\sigma$ and $a \in (\delta^{\mathcal{Q}})^{<\omega}$.

(3) above uses the fact that Θ is regular.

Let $\delta = \delta_\alpha^{\mathcal{H}^*_\sigma}$ be a Woodin cardinal of $\mathcal{H}^*_\sigma$ such that $\rho_\omega(\mathcal{H}^*_\sigma) \leq \delta$. Let $A \subseteq \delta$ witness this. So A is a bounded subset of θ^σ that is not in $\mathcal{H}^*_\sigma$. We aim to obtain a contradiction from this.

Now we can construe $(\mathcal{H}^*_\sigma, \Sigma_\sigma)$ as a $(\mathcal{H}_\sigma(\alpha), \Sigma_{\mathcal{H}_\sigma(\alpha)})$-hod pair. We can define a direct limit system of $(\mathcal{H}_\sigma(\alpha), \Sigma_{\mathcal{H}_\sigma(\alpha)})$ hod pairs as follows:

$$\mathcal{F}^* = \{(\mathcal{Q}', \Lambda') \mid (\mathcal{Q}', \Lambda') \equiv_{DJ} (\mathcal{Q}, \Lambda)\}\ [13].$$

Note that $\mathcal{F}$ does not depend on $(\mathcal{Q}, \Lambda)$ and in fact is $OD_{\Sigma_{\mathcal{H}_\sigma(\alpha)}}$ in Γ. This easily implies that A is $OD_{\Sigma_{\mathcal{H}_\sigma(\alpha)}}$. By $\mathsf{MC}(\Sigma_{\mathcal{H}_\sigma(\alpha)})$ [14] and the fact that $\mathcal{H}_\sigma(\alpha+1)$ is $\Sigma_{\mathcal{H}_\sigma(\alpha)}$-full, $A \in \mathcal{H}_\sigma(\alpha+1) \in \mathcal{H}^*_\sigma$. This contradicts the definition of A. □

We define a measure ν on Θ over $\mathcal{H}^+$ as follows. Let $A \in \mathcal{H}^+$. Then

$$A \in \nu \Leftrightarrow \forall^*_{\mu_N}\sigma\ \sup(\sigma \cap \Theta) \in A. \tag{5.1}$$

The definition makes sense since $\mathrm{cof}(\Theta) > \omega$. It's clear that ν is a measure. Note also that the above definition makes sense for all $A \in V$ but we only care about those A's in $\mathcal{H}^+$ as we can prove the measure behaves nicely on this collection of sets.

First we note that $\mathcal{H}^+$ is a ZFC^- model. Now we show the following.

Lemma 5.4. *ν is amenable to $\mathcal{H}^+$. In other words, for any $\mathcal{M} \lhd \mathcal{H}^+$, $\nu \restriction \mathcal{M} \in \mathcal{H}^+$.*

[13] This means these $(\mathcal{H}^*_\sigma, \Sigma_\sigma)$ hod pairs are Dodd-Jensen equivalent.

[14] This stands for Mouse Capturing with respect to $\Sigma_{\mathcal{H}_\sigma(\alpha)}$, which in turns is the statement that if $x, y \in \mathbb{R}$, and x is $OD_{\Sigma_{\mathcal{H}_\sigma(\alpha)}}(y)$ then x is in a $\Sigma_{\mathcal{H}_\sigma(\alpha)}$-mouse over y.

Proof. Let $\mathcal{M} \lhd \mathcal{H}^+$ be sound and $\rho_\omega(\mathcal{M}) \leq \Theta$ (note that $\mathcal{H}^+$ is the union of such $\mathcal{M}$'s). Let $\nu_\mathcal{M} = \nu \restriction \mathcal{M}$. We show $\nu_\mathcal{M} \in \mathcal{H}^+$.

Again, we fix N as above and assume that $\mathcal{M} \in N$. We use the set-up and notations above. Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_{\mu_N}$. Note that $\forall^*_{\mu_N}\sigma\ \mathcal{M}_\sigma = \pi_\sigma^{-1}(\mathcal{M})$.

We want to show $\forall^*_{\mu_N}\sigma\ \nu_\sigma \in \mathcal{H}^+_\sigma$. Fix such a σ, let $\mathcal{R}_\sigma = \mathrm{HOD}_{(\mathcal{H}^+_\sigma, \Sigma^-_\sigma)}$. Note that $\wp(\Theta_\sigma) \cap \mathcal{R}_\sigma = \wp(\Theta_\sigma) \cap \mathcal{H}^+_\sigma$ by the similar argument to that used in Lemma 5.2. Let $\vec{A} = \langle A_\alpha \mid \alpha < \Theta_\sigma\rangle$ be a definable-over-$\mathcal{M}_\sigma$ enumeration of $\wp(\Theta_\sigma) \cap \mathcal{M}_\sigma$. We want to show $\langle \alpha \mid A_\alpha \in \nu_\sigma\rangle \in \mathcal{R}_\sigma$ which in turns implies $\langle \alpha \mid A_\alpha \in \nu_\sigma\rangle \in \mathcal{H}^+_\sigma$.

Let $\gamma_\sigma = \sup(\pi_\sigma[\Theta_\sigma])$ (note that $\pi_\sigma[\Theta_\sigma] = \sigma \cap \Theta$ coincides with the iteration embedding via Σ^-_σ and since $\mathrm{cof}(\Theta) > \omega$, $\gamma_\sigma < \Theta$). Note that

$$\forall \alpha < \Theta_\sigma\ A_\alpha \in \nu_\sigma \Leftrightarrow \gamma_\sigma \in \pi_\sigma(A) \cap (\gamma_\sigma + 1) \tag{5.2}$$

and

$$\langle \pi_\sigma(A_\alpha) \cap (\gamma_\sigma + 1) \mid \alpha < \Theta_\sigma\rangle \in \mathcal{R}_\sigma. \tag{5.3}$$

5.3 is true because $\langle \pi_\sigma(A_\alpha) \mid \alpha < \Theta_\sigma\rangle \in \mathcal{H}^+$. Hence $\langle \pi_\sigma(A_\alpha) \cap (\gamma_\sigma + 1) \mid \alpha < \Theta_\sigma\rangle \in \mathcal{H}^+|\Theta$. Since $\mathcal{H}^+ \subseteq \mathcal{R}_\sigma$, we have 5.3.

By Equations 5.2 and 5.3, we have $\langle \alpha \mid A_\alpha \in \nu_\sigma\rangle \in \mathcal{R}_\sigma$. This completes the proof of the lemma. $\square$

Remark 5.5. *(i) In the proof of Lemma 5.4, we can't demand that $\mathcal{H}^+ \in N$ because it may be the case that $o(\mathcal{H}^+) = \Theta^+$ and hence there are no surjections from $\wp(\mathbb{R})$ onto $\mathcal{H}^+$.*

(ii) It follows from the fact that Θ is regular and $\mathsf{AD}_\mathbb{R}$ holds that $\mathcal{H}^+ \vDash$ "Θ is regular limit of Woodin cardinals".

Now we want to show that ν is normal and $\wp(\Theta) \cap L[\mathcal{H}^+, \nu] = \wp(\Theta) \cap \mathcal{H}^+$. Let $\mathcal{M} \lhd \mathcal{H}^+$ be sound and $\rho_\omega(\mathcal{M}) \leq \Theta$.

Lemma 5.6. *$\nu_\mathcal{M} =_{def} \nu \restriction \mathcal{M}$ is normal.*

Proof. Suppose not. Let N be as above and as before, we assume that $\mathcal{M}, \nu_\mathcal{M} \in N$. We reuse the notation and objects defined above for N. Let $\mathcal{M} = [\sigma \mapsto \mathcal{M}_\sigma]_{\mu_N}$ and note that $\forall^*_{\mu_N}\sigma\ \mathcal{M}_\sigma = \pi_\sigma^{-1}(\mathcal{M})$.

We define a measure ν_σ on Θ_σ over $\mathcal{M}_\sigma$ as follows.

$$A \in \nu_\sigma \Leftrightarrow \gamma_\sigma =_{\mathrm{def}} \sup(\pi_\sigma[\Theta_\sigma]) \in \pi_\sigma(A). \tag{5.4}$$

It's easy to see that

$$\nu_\sigma = \pi_\sigma^{-1}(\nu_\mathcal{M}) \wedge \Pi_\sigma \nu_\sigma/\mu_N = \nu_\mathcal{M}. \tag{5.5}$$

By the assumption on $\nu_\mathcal{M}$, we have that $\forall^*_{\mu_N}\sigma\ \nu_\sigma$ is not normal. This means

$$\{\pi_\sigma(f)(\gamma_\sigma) \mid f \in \mathcal{M}_\sigma\} \cap \gamma_\sigma \neq \sigma \cap \gamma_\sigma. \tag{5.6}$$

In other words, by normality of μ_N,

$$\exists f \in \mathcal{M}\ \forall^*_{\mu_N}\sigma\ f(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma \wedge f(\gamma_\sigma) < \gamma_\sigma.$$

Fix such an $f \in \mathcal{M}$ and let

$$A' = \{\sigma \mid f(\gamma_\sigma) \notin \sigma \cap \gamma_\sigma \wedge f(\gamma_\sigma) < \gamma_\sigma\}. \tag{5.7}$$

We have $A' \in \mu_N$. This implies that $B \in \nu_{\mathcal{M}}$ where

$$B = \{\gamma \mid f(\gamma) < \gamma\}. \tag{5.8}$$

Let $\mathcal{M} \lhd \mathcal{M}^* \lhd \mathcal{H}^+$ be such that $\nu_{\mathcal{M}} \in \mathcal{M}^*$. This is possible since $\nu_{\mathcal{M}} \in \mathcal{H}^+$ and $\mathcal{H}^+$ is a limit of such $\mathcal{M}^*$'s. Now we can also assume $\mathcal{M}^* \in N$ by expanding N if necessary. Let then $\forall^*_{\mu_N}\sigma\ \mathcal{M}^*_\sigma = \pi_\sigma^{-1}(\mathcal{M}^*)$.

Claim 5.7. *There is an $\eta < \Theta$ such that $\forall^*_{\mu_N}\sigma\ f(\gamma_\sigma) = \eta$.*

Proof. $\forall^*_{\mu_N}\sigma$, let Σ_σ be the π_σ-guided strategy for $\mathcal{M}_\sigma$ (below Θ_σ as defined in the proof of Lemma 5.2) and $i_\sigma : \mathcal{M}_\sigma \to \mathcal{N}_\sigma$ be the direct map, where $\mathcal{N}_\sigma$ is the direct limit of all Σ_σ-iterates of $\mathcal{M}_\sigma$. Note that since $\mathcal{M}_\sigma \vDash$ "Θ_σ is regular", $i_\sigma \restriction \Theta_\sigma = \pi_\sigma \restriction \Theta_\sigma$; also i_σ is cofinal in $\mathrm{o}(\mathcal{N}_\sigma)$. These properties follow from (1)-(4) in the proof of Lemma 5.2. (1)-(4) in the proof of Lemma 5.2 also imply that there is a map $k_\sigma : \mathcal{N}_\sigma \to \mathcal{M}$ such that $k_\sigma \circ i_\sigma = \pi_\sigma \restriction \mathcal{M}_\sigma$ and $\mathrm{crt}(k_\sigma) = i_\sigma(\Theta_\sigma) = \gamma_\sigma$.

Let $\nu^*_\sigma = i_\sigma[\nu_\sigma]$ and $(f_\sigma, B_\sigma) = (\pi_\sigma^{-1}(f), \pi_\sigma^{-1}(B))$. We have then that $\forall^*_{\mu_N}\sigma\ B_\sigma \in \nu_\sigma$, which implies that $i_\sigma(B_\sigma) \in \nu^*_\sigma$. We note that ν^*_σ is normal; this is because ν^*_σ is induced from k_σ and $\mathrm{crt}(k_\sigma) = \gamma_\sigma$.

To prove the lemma, it suffices to show that

$$\forall^*_{\mu_N}\sigma\ \mathcal{M}^*_\sigma \vDash \exists \eta_\sigma < \Theta_\sigma\ i_{\nu_\sigma}(f_\sigma)(\Theta_\sigma) = \eta_\sigma. \tag{5.9}$$

Fix a σ in the first paragraph. Note that we can extend i_σ to a map $i^+_\sigma : \mathcal{M}^*_\sigma \to \mathcal{N}^*_\sigma$ such that $i^+_\sigma \restriction \Theta_\sigma = i_\sigma \restriction \Theta_\sigma = \pi_\sigma \restriction \Theta_\sigma$ and extend k_σ to a map $k^+_\sigma : \mathcal{N}^*_\sigma \to \mathcal{M}^*$ such that $\mathrm{crt}(k^+_\sigma) = \mathrm{crt}(k_\sigma) = \gamma_\sigma$ and $k^+_\sigma \restriction \mathcal{N}_\sigma = i_\sigma$.

As mentioned above, the measure $\nu^*_\sigma \in \mathcal{N}^*_\sigma$ is normal so

$$\mathcal{N}^*_\sigma \vDash \exists \eta < \gamma_\sigma\ i_{\nu^*_\sigma}(i_\sigma(f))(\gamma_\sigma) = \eta \tag{5.10}$$

By 5.10 and elementarity, 5.9 holds for σ. □

The conclusion of the lemma follows from the previous claim. □

Theorem 5.8. *Let $\mathcal{H}^{++} = \mathcal{H}^{+-}[\nu]$.*[15] *Then $\wp(\Theta) \cap \mathcal{H}^{++} = \wp(\Theta) \cap \mathcal{H}^+$.*

[15] We identify ν with the total extender over $\mathcal{H}^+$ that has index Θ^{++} of $\mathcal{H}^{+-}$.

Proof. Suppose not. Then there is an $\mathcal{M}^* \trianglelefteq \mathcal{H}^{++}$ such that $\rho(\mathcal{M}^*) \leq \Theta$ and $\mathcal{M}^*$ defines a set not in $\mathcal{H}^+$. We may assume $\mathcal{M}^*$ is minimal and $\rho_1(\mathcal{M}^*) \leq \Theta$ (note that $\mathrm{o}(\mathcal{M}^*) \geq (\Theta^{++})^{\mathcal{H}^{+-}}$). Let $\mathcal{M}$ be the transitive collapse of $Hull_1^{\mathcal{M}^*}(\Theta \cup p_1^{\mathcal{M}^*})$. Then $\mathcal{M}$ is transitive and $\mathcal{M}$ Σ_1-defines a set not in $\mathcal{H}^+$; so $\mathcal{M}$ has the form $L_\alpha[\mathcal{H}^*][\nu_{\mathcal{M}}]$ for some $\mathcal{H}^*, \nu_{\mathcal{M}}$. It's easy to see that $\nu_{\mathcal{M}} = \nu \restriction \mathcal{M}$.

Let $N \vDash \mathsf{ZF}^- + \mathsf{DC}$ be transitive such that $\wp(\mathbb{R}) \twoheadrightarrow N$ and $\mathcal{M}, \Gamma, \nu_{\mathcal{M}} \in N$. Let μ_N be the supercompact measure on $\wp_{\omega_1}(N)$ induced by μ_Γ. $\forall^*_{\mu_N}\sigma$, let $\pi_\sigma : M_\sigma \to N$ be the uncollapse map. Let $\pi_\sigma(\mathcal{M}_\sigma, \mathcal{H}_\sigma, \Theta_\sigma, \nu_\sigma) = (\mathcal{M}, \mathcal{H}, \Theta, \nu_{\mathcal{M}})$.

Lemma 5.9. *For μ_N-almost-all σ, there is an iteration strategy Σ_σ^+ for $\mathcal{M}_\sigma$ with the following properties:*

1. *Σ_σ^+ is a π_σ-realizable strategy that extends Σ_σ. This means $\Sigma_\sigma \subseteq \Sigma_\sigma^+$ and whenever $\vec{\mathcal{T}}$ is a stack according to Σ_σ^+, letting $i : \mathcal{M}_\sigma \to \mathcal{P}$ be the iteration embedding, then there is a map $k : \mathcal{P} \to \mathcal{M}$ such that $\pi_\sigma = k \circ i$.*

2. *Whenever $(\mathcal{Q}, \Lambda) \in I(\mathcal{M}_\sigma, \Sigma_\sigma^+)$, $\forall \alpha < \lambda^{\mathcal{Q}}$, $\Lambda_{\mathcal{Q}(\alpha)}$ is $\Gamma(\mathcal{M}_\sigma, \Sigma_\sigma^+)$-fullness preserving and has branch condensation. Hence Σ_σ^+ is $\Gamma(\mathcal{M}_\sigma, \Sigma_\sigma^+)$-fullness preserving.*

Proof. We prove (1) (see Diagram 1). The proof of (2) is just the proof of [5, Theorem 3.26] so we omit it; we just mention the key point in proving (2) is that $\Lambda_{\mathcal{Q}(\alpha)}$ for $\alpha < \lambda^{\mathcal{Q}}$ is a pullback of a strategy that is fullness preserving and has branch condensation.

Fix a σ. Suppose $i : (\mathcal{M}_\sigma, \nu_\sigma) \to (\mathcal{P}, \nu_{\mathcal{P}})$ is the ultrapower map using ν_σ. We describe how to obtain a π_σ-realizable strategy $\Sigma_{\mathcal{P}(\alpha)}$ for $\alpha < \lambda^{\mathcal{P}}$. We then let $\Sigma_{\mathcal{P}}^- = \oplus_{\alpha<\lambda^{\mathcal{P}}} \Sigma_{\mathcal{P}(\alpha)}$ and $\vec{\mathcal{T}}$ be a stack on $\mathcal{P}$ according to $\Sigma_{\mathcal{P}}^-$ with end model $\mathcal{Q}$. Let $j : (\mathcal{P}, \nu_{\mathcal{P}}) \to (\mathcal{Q}, \nu_{\mathcal{Q}})$ be the iteration map and $k : \mathcal{Q} \to \mathcal{R}$ be the ultrapower map by $\nu_{\mathcal{Q}}$. We describe how to obtain π_σ-realizable strategy $\Sigma_{\mathcal{Q}(\alpha)}$ for all $\alpha < \lambda^{\mathcal{Q}}$ and a π_σ-realizable strategy $\Sigma_{\mathcal{R}(\alpha)}$ for all $\alpha < \lambda^{\mathcal{R}}$. The construction of the strategy for this special case has all the ideas needed to construct the full strategy as for the general stack (in normal form), we simply repeat the arguments given below inductively.

Let $\tau \prec N$ be such that $\sigma, \vec{\mathcal{T}} \in \tau$ and are countable there. μ_N-allmost-all τ have this property. Let $\pi_{\sigma,\tau} = \pi_\tau^{-1} \circ \pi_\sigma$. Working in N_τ, $\gamma_0 = i^{\Sigma_\sigma^-}_{\mathcal{H}_\sigma,\infty}(\lambda^{\mathcal{M}_\sigma})$. Let $i^* : \mathcal{P} \to \mathcal{M}_\tau$ be such that

$$i^*(i(f)(\lambda^{\mathcal{M}_\sigma})) = \pi_{\sigma,\tau}(f)(\gamma_0).$$

By the definition of ν_σ, it's not hard to show i^* is elementary and $\pi_{\sigma,\tau} = i^* \circ i$ (so $\pi_\sigma = \pi_\tau \circ i^* \circ i$).

Note also that $i^*(\nu_{\mathcal{P}}) = \nu_\tau$. Now, let $(\mathcal{N}, \Lambda)$ be a point in the direct limit system giving rise to $\mathcal{H}_\tau$ be such that $ran(i^* \restriction \lambda^{\mathcal{P}}) \subseteq ran(i^\Lambda_{\mathcal{N},\infty})$. There is some $s : \mathcal{P}|\lambda^{\mathcal{P}} \to \mathcal{N}$ such that $i^\Lambda_{\mathcal{N},\infty} \circ s = i^* \restriction \lambda^{\mathcal{P}}$. Then $\Sigma_{\mathcal{P}}^-$ is simply the s-pullback of Λ. Note that Λ can be extended to a fullness preserving strategy with branch condensation. It's not hard to show that the definition of $\Sigma_{\mathcal{P}}^-$ doesn't depend on the choice of $(\mathcal{N}, \Lambda)$ and the choice of τ. We show why $\Sigma_{\mathcal{P}}^-$ doesn't depend on the choice of $(\mathcal{N}, \Lambda)$. Suppose $(\mathcal{N}, \Lambda)$, $(\mathcal{N}', \Lambda')$, $s : \mathcal{P}|\lambda^{\mathcal{P}} \to \mathcal{N}$, and $s' : \mathcal{P}|\lambda^{\mathcal{P}} \to \mathcal{N}'$ are as in the definition of $\Sigma_{\mathcal{P}}^-$, then we can compare $(\mathcal{N}, \Lambda), (\mathcal{N}', \Lambda')$ and get a common iterate $(\mathcal{S}, \Psi)$, where Ψ is the common tail of Λ and Λ'. Let $i_{\mathcal{N},\mathcal{S}} : \mathcal{N} \to \mathcal{S}$ and $i_{\mathcal{N}',\mathcal{S}} : \mathcal{N}' \to \mathcal{S}$ be iteration maps. Note that $i_{\mathcal{N},\mathcal{S}} \circ s = i_{\mathcal{N}',\mathcal{S}} \circ s' =_{\mathrm{def}} t$ and so

$$\begin{tikzcd}
& & & \mathcal{M}, \nu_{\mathcal{M}} \\
& & \mathcal{M}_\tau, \nu_\tau \arrow[r, "h"] \arrow[ur, "\pi_\tau"] & Ult(\mathcal{M}_\tau, \nu_\tau) \arrow[u, "h^*"'] \\
\mathcal{M}_\sigma, \nu_\sigma \arrow[r, "i"'] \arrow[urr, "\pi_{\sigma,\tau}"] & \mathcal{P}, \nu_{\mathcal{P}} \arrow[r, "\vec{\mathcal{T}}, j"'] \arrow[ur, "i^*"'] & \mathcal{Q}, \nu_{\mathcal{Q}} \arrow[r, "k"'] \arrow[u, "j^*"] & \mathcal{R}, \nu_{\mathcal{R}} \arrow[u, "k^*"']
\end{tikzcd}$$

Figure 1: The construction of Σ_σ^+

$$\Lambda^s = (\Lambda')^{s'} = \Psi^t.$$

A similar argument shows that $\Sigma^-_{\mathcal{P}}$ does not depend on the choice of τ. Let $\mathcal{P}_\infty$ be the direct limit of $\Sigma^-_{\mathcal{P}}$ iterates of $\mathcal{P}|\delta^{\mathcal{P}}$ and $\pi_{\mathcal{P}} : \mathcal{P}_\infty \to \mathcal{H}_\tau$ be the natural map such that $\pi_{\mathcal{P}} \circ i^{\Sigma^-_{\mathcal{P}}}_{\mathcal{P}|\delta^{\mathcal{P}},\infty} = i^* \restriction (\mathcal{P}|\delta^{\mathcal{P}})$.

Now every element of $\mathcal{Q}$ has the form $j(f)(a)$ for some $f \in \mathcal{P}$ and $a \in \alpha(\vec{\mathcal{T}})^{<\omega}$. We let $j^* : \mathcal{Q} \to \mathcal{M}_\tau$ be such that $j^*(j(f)(a)) = i^*(f)(\pi_{\mathcal{P}}(i^{\Sigma^-_{\mathcal{Q}}}_{\mathcal{Q},\infty}(a)))$. Hence $i^* = j^* \circ j$ and $\pi_\sigma = j^* \circ j \circ i$.

Finally, every element of $\mathcal{R}$ has the form $k(f)(\lambda^{\mathcal{Q}})$ for some $f \in \mathcal{Q}$. Let $h : \mathcal{M}_\tau \to Ult(\mathcal{M}_\tau, \nu_\tau)$ be the ultrapower map and $h^* : Ult(\mathcal{M}_\tau, \nu_\tau) \to \mathcal{M}$ be such that $\pi_\tau = h^* \circ h$. Then let $k^* : \mathcal{Q} \to Ult(\mathcal{M}_\tau, \nu_\tau)$ be such that $k^*(k(f)(\lambda^{\mathcal{Q}})) = h(j^*(f))(\lambda^{\mathcal{M}_\tau})$. It's easy to see that $h \circ j^* = k^* \circ k$. We can now derive the strategy $\Sigma^-_{\mathcal{R}}$ using $h^* \circ k^* \restriction \lambda^{\mathcal{R}}$ the same way we used $i^* \restriction \lambda^{\mathcal{P}}$ to derive the strategy $\Sigma^-_{\mathcal{P}}$. Again, it's easy to show that $\Sigma^-_{\mathcal{R}}$ is a π_σ-realizable strategy. The definition of $\Sigma^-_{\mathcal{R}}$ does not depend on the choice of τ.

In general, suppose $\vec{\mathcal{T}} = (\mathcal{T}_\alpha, \mathcal{N}_\beta : \alpha < \gamma, \beta \leq \gamma)$ is a countable stack on $\mathcal{M}_\sigma$ in normal form according to Σ^+_σ and $\mathcal{T}_\gamma$ is on $\mathcal{N}_\gamma$. We want to define Σ^+_σ on $\mathcal{T}_\gamma$. As part of the definition of Σ^+_σ, we have iteration map $i_{\mathcal{M}_\sigma,\mathcal{N}_\alpha} : (\mathcal{M}_\sigma, \nu_\sigma) \to (\mathcal{N}_\alpha, \nu_\alpha)$, a map $i : (\mathcal{N}_\alpha, \nu_\alpha) \to (\mathcal{M}_\tau, \nu_\tau)$ for a sufficiently large τ that contains all relevant objects, i-pullback strategy Σ_α for $\mathcal{N}_\alpha|\delta^{\mathcal{N}_\alpha}$. If $\mathcal{T}_\gamma = \langle \mathcal{N}_\alpha, \nu_\alpha \rangle$, then we can define maps $k^* : \mathrm{Ult}(\mathcal{N}_\alpha, \nu_\alpha) \to \mathrm{Ult}(\mathcal{M}_\tau, \nu_\tau)$, $h : \mathcal{M}_\tau \to \mathrm{Ult}(\mathcal{M}_\tau, \nu_\tau)$, and $h^* : \mathrm{Ult}(\mathcal{M}_\tau, \nu_\tau) \to \mathcal{M}$ as above and derive a strategy $\Sigma_{\alpha+1}$ for $\mathcal{N}_{\alpha+1}|\delta^{\mathcal{N}_\alpha+1}$, where $\mathcal{N}_{\alpha+1} = \mathrm{Ult}(\mathcal{N}_\alpha, \nu_\alpha)$. We then let $\Sigma_{\alpha+1} \subset \Sigma^+_\sigma$. Suppose $\mathcal{T}_\gamma$ is below $\delta^{\mathcal{N}_\alpha}$. Then we use $\Sigma_\alpha \subset \Sigma^+_\sigma$ to choose a branch b for $\mathcal{T}_\gamma$ and a map $j^* : \mathcal{N}^{\mathcal{T}^\frown b} \to \mathcal{M}_\tau$ such that $j^* \circ i^{\mathcal{T}}_b = i_\alpha$.

This completes the construction of Σ^+_σ and hence the proof of Lemma 5.9. Note it also follows that Σ^+_σ extends Σ_σ.

$\square$

By a ZFC-comparison argument ([5, Section 2.7]) and the fact that Σ^+_σ is $\Gamma(\mathcal{M}_\sigma, \Sigma^+)$-fullness preserving, an iterate of Σ^+_σ has branch condensation. Without loss of generality, we may assume Σ^+_σ has branch condensation.

Since $\rho_1(\mathcal{M}_\sigma) \leq \Theta_\sigma$, we let $A \subseteq \Theta_\sigma$ be a set Σ_1 definable over $\mathcal{M}_\sigma$ but not in $\mathcal{H}_\sigma^+$. Say

$$\alpha \in A \Leftrightarrow \mathcal{M}_\sigma \vDash \psi[\alpha, s, p_1^{\mathcal{M}_\sigma}], \tag{5.11}$$

for some $s \in \Theta_\sigma^{<\omega}$. Recall that $\mathcal{M}_\sigma \vDash \Theta_\sigma$ is measurable as witnessed by ν_σ. We can define a direct limit system $\mathcal{F} = \{(\mathcal{Q}, \Lambda) \mid (\mathcal{Q}, \Lambda) \equiv_{DJ} (\mathcal{M}_\sigma, \Sigma_\sigma^+)\}$ [16]. Let $\mathcal{M}_\infty$ be the direct limit of $\mathcal{F}$ and let $i_{\mathcal{M}_\sigma,\infty} : \mathcal{M}_\sigma \to \mathcal{M}_\infty$ be the iteration embedding. We have that $\mathrm{HOD}|\gamma_\sigma \lhd \mathcal{M}_\infty \in \mathrm{HOD}$ and $\rho_1(\mathcal{M}_\infty) \leq \gamma_\sigma$. Let A_∞ be defined over $\mathcal{M}_\infty$ the same way A is defined over $\mathcal{M}_\sigma$, i.e.

$$\alpha \in A_\infty \Leftrightarrow \mathcal{M}_\infty \vDash \psi[\alpha, i_{\mathcal{M}_\sigma,\infty}(s), p_1^{\mathcal{M}_\infty}]. \tag{5.12}$$

Since A_∞ is OD, A is ordinal definable from $(\mathcal{H}_\sigma, \Sigma_\sigma^-)$. By SMC, $A \in \mathcal{H}_\sigma^+$. Contradiction. $\square$

Lemma 5.10. *$L[\mathcal{H}^{++}](\Gamma) \cap \wp(\mathbb{R}) = \Gamma$ and $L[\mathcal{H}^{++}](\Gamma) \vDash \mathsf{AD}_\mathbb{R} +$ there is an $\mathbb{R}$-complete measure on Θ.*

Proof. First note that no $\mathcal{H}^{++} \lhd \mathcal{M} \lhd L[\mathcal{H}^{++}]$ is such that $\rho_\omega(\mathcal{M}) \leq \Theta$. The equality of in the conclusion of the lemma follows from Theorem 4.1 with $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ playing the role of $\mathcal{H}$ and $L[\mathcal{H}^{++}]$ playing the role of $\mathcal{H}^+$. Note that $L[\mathcal{H}^{++}] \vDash$ "Θ is regular". Hence we get $L[\mathcal{H}^{++}](\Gamma) \vDash$ "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular" by Lemma 4.5. The $\mathbb{R}$-complete measure on Θ in $L[\mathcal{H}^{++}](\Gamma)$ comes from ν from the proof of Theorem 2.4 in [2]. The proof uses the fact that every $A \in \Gamma$ can be added to $L[\mathcal{H}^{++}]$ via a forcing of size $< \Theta$. This means every $A \subseteq \Theta$ in $L[\mathcal{H}^{++}](\Gamma)$ is in some generic extension of $L[\mathcal{H}^{++}]$ via a forcing of size $< \Theta$ and hence is measured by the canonical extension of ν. The $\mathbb{R}$-completeness of the induced measure then follows from [2, Theorem 2.4]. $\square$

This completes the proof of Theorem 1.6.

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