HOD in Natural Models of AD^+

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Abstract

The goal of this paper is to compute the full HOD of models of AD^+ of the form $L(\wp(\mathbb{R}))$ below "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". As part of this computation, we give a computation of HOD$|\Theta$ left open in [3] for Θ a successor in the Solovay sequence. Our work, when combined with [3], shows that in AD^+ models of the form $L(\wp(\mathbb{R}))$ below "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular", HOD satisfies GCH.

1 Introduction

Throughout this paper, unless stated otherwise, we assume

$$V = L(\wp(\mathbb{R})) + \mathsf{AD}^+ + \text{no } \mathsf{AD}^+ \text{ model } M \text{ containing } \mathbb{R}\cup\mathrm{OR}$$
$$\text{satisfies "}\mathsf{AD}_{\mathbb{R}} + \Theta \text{ is regular"}^1.$$

We call this assumption $(*)$. Under the smallness assumption $(*)$, we analyze full HOD, extending the analysis in [3]. Our smallness assumption is made because of the fact that for our computation, we rely heavily on the theory of hod mice, which is developed in [3] for models satisfying the assumption[2].

To put this work in a proper context, we recall a bit of history on the computation of HOD. In $L(\mathbb{R})$ under AD[3], Harrington and Kechris show that HOD $\vDash \mathsf{CH}$. Let $\kappa = \omega_1^{L(\mathbb{R})}$. Solovay shows that HOD $\vDash \kappa$ is measurable and Becker shows κ is the least measurable

[1] Under our hypothesis, "Strong Mouse Capturing" (SMC) holds. This notion will be introduced in Section 2.

[2] Recently, there has been significant progress made in understanding HOD of AD^+ models up to the theory "$\Theta = \theta_{\alpha+1}$, where θ_α is the largest Suslin cardinal" (cf. [4]).

[3] It's known that if $L(\mathbb{R}) \vDash \mathsf{AD}$ then $L(\mathbb{R}) \vDash \mathsf{AD}^+$.

in HOD. These were shown using descriptive set theory. Then Steel in [15] or [12] using inner model theory shows HOD$|\Theta$ is a fine-structural mouse, which in particular implies $V_\Theta^{\mathrm{HOD}} \vDash \mathsf{GCH}$. Woodin (see [11]), building on Steel's work, completes the full HOD analysis in $L(\mathbb{R})$ and shows HOD $\vDash$ GCH and furthermore shows that the full HOD of $L(\mathbb{R})$ is a hybrid mouse that contains some information about a certain iteration strategy of its initial segments. A key fact used in the computation of HOD in $L(\mathbb{R})$ is that if $L(\mathbb{R}) \vDash \mathsf{AD}$ then $L(\mathbb{R}) \vDash \mathsf{MC}$[4]. It's natural to ask whether analogous results hold in the context of $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$. The HOD computation is an integral part of the structural analysis of AD^+ models and plays an important role in applications such as the core model induction. Woodin has shown that under this assumption HOD $\vDash \mathsf{CH}$. Recently, Grigor Sargsyan in [3], assuming $(*)$, proves Strong Mouse Capturing (SMC) (a generalization of MC) and computes HOD$|\Theta$ for Θ being limit in the Solovay sequence and HOD$|\theta_\alpha$ for $\Theta = \theta_{\alpha+1}$ in a similar sense as above under the assumption $(*)$.

This paper extends work of Steel, Woodin, and Sargsyan to the computation of full HOD under $(*)$. There are two main cases. The case Θ is a limit in the Solovay sequence (see Section 2), i.e. $\Theta = \theta_\alpha$ for some limit α, is dealt with in Section 3. There the HOD computation is split into two cases depending on whether or not HOD $\vDash \mathrm{cof}(\Theta)$ is measurable. Let $\mathcal{M}_\infty, \mathcal{M}_\infty^+, \mathcal{N}_\infty^+$ be defined as in Section 3, we prove the following theorems.

Theorem 1.1. *Assume $(*)$. Suppose $V \vDash$ "$\Theta = \theta_\alpha$ for some limit α" and HOD $\vDash$ "$cof(\Theta)$ is not measurable". Then HOD $= L[\mathcal{M}_\infty]$.*

Theorem 1.2. *Assume $(*)$. Suppose $V \vDash$ "$\Theta = \theta_\alpha$ for some limit α" and HOD $\vDash$ "$cof(\Theta)$ is measurable". Then*

1. *HOD $= L[\mathcal{M}_\infty, \mathcal{M}_\infty^+]$.*
2. *HOD $= L[\mathcal{N}_\infty^+]$.*

Theorem 1.1 and (2) of Theorem 1.2 imply that HOD is a hybrid (fine-structural) premouse. In Section 4, assuming $(*)$, we compute full HOD in the case $\Theta = \theta_0$ or Θ is a successor in the Solovay sequence. Let $\mathcal{M}_\infty, \Sigma_\infty$ be defined as in Section 4. Roughly, $\mathcal{M}_\infty$ is a certain direct limit extending HOD$|\Theta$ and Σ_∞ is a fragment of the strategy for $\mathcal{M}_\infty|\Theta$ on (finite stacks of) normal trees in $\mathcal{M}_\infty$. We outline the proof of the following theorem.

Theorem 1.3. *Assume $(*)$. Suppose $V \vDash \Theta = \theta_0$ or $V \vDash \Theta = \theta_{\alpha+1}$ for some α. Then HOD $= L[\mathcal{M}_\infty][\Sigma_\infty]$.*

[4] MC stands for Mouse Capturing, which is the statement that if $x, y \in \mathbb{R}$, then $x \in OD(y) \Leftrightarrow x$ is in a mouse over y.

The reader familiar with the HOD computation in $L(\mathbb{R})$ will not be surprised here. In fact, this is where most of the main ideas in Section 4 come from. However, to make these ideas work, we have to bring in the theory of hod mice from [3] and recent results on capturing sets of reals by Σ-mice over $\mathbb{R}$ from [5].

Our work combined with Sargsyan's work in [3] show that:

Corollary 1.4. *Assume* $(*)$. *HOD* $\vDash$ *GCH.*

The question of whether HOD of an arbitrary AD^+ model satisfies GCH still remains open and is one of the central open problems in descriptive inner model theory.

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2 Backgrounds

In this section, we recall some basic facts about AD^+, hod mice, and a certain kind of Prikry forcing. The reader familiar with this material can skip to Section 3.

2.1 Basic facts about AD^+ and hod mice

We start with the definition of Woodin's theory of AD^+. In this paper, we identify $\mathbb{R}$ with ω^ω. We use Θ to denote the sup of ordinals α such that there is a surjection $\pi : \mathbb{R} \to \alpha$.

Definition 2.1. *AD^+ is the theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ and*

1. *for every set of reals A, there are a set of ordinals S and a formula φ such that $x \in A \Leftrightarrow L[S, x] \vDash \varphi[S, x]$. (S, φ) is called an ∞-Borel code for A;*

2. *for every $\lambda < \Theta$, for every continuous $\pi : \lambda^\omega \to \omega^\omega$, for every $A \subseteq \mathbb{R}$, the set $\pi^{-1}[A]$ is determined.*

AD^+ is known to be equivalent to "AD + the set of Suslin cardinals is closed" (see [1]). Another, perhaps more useful, equivalence of AD^+ is "$\mathsf{AD} + \Sigma_1$ statements reflect to the Suslin-co-Suslin sets" (see [8] for a more precise statement).

Definition 2.2 (AD^+)**.** *The* ***Solovay sequence*** *is the sequence $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ where*

1. *θ_0 is the sup of ordinals β such that there is an OD surjection from $\mathbb{R}$ onto β;*

2. *if $\alpha > 0$ is limit, then $\theta_\alpha = \sup\{\theta_\beta \mid \beta < \alpha\}$;*

3. *if $\alpha = \beta + 1$ and $\theta_\beta < \Theta$ (i.e. $\beta < \Omega$), fixing a set $A \subseteq \mathbb{R}$ of Wadge rank θ_β, θ_α is the sup of ordinals γ such that there is an $OD(A)$ surjection from $\mathbb{R}$ onto γ, i.e. $\theta_\alpha = \theta_A$.*

Note that the definition of θ_α for $\alpha = \beta + 1$ in Definition 2.2 does not depend on the choice of A. We recall some basic notions from descriptive set theory.

Suppose $A \subseteq \mathbb{R}$ and (N, Σ) is such that N is a transitive model of "$\mathsf{ZFC} -$ *Replacement*" and Σ is an (ω_1, ω_1)-iteration strategy or just ω_1-iteration strategy for N. We use $o(N)$, OR^N, ORD^N interchangably to denote the ordinal height of N. Suppose that δ is countable in V but is an uncountable cardinal of N and suppose that $T, U \in N$ are trees on $\omega \times (\delta^+)^N$. We say (T, U) *locally Suslin captures A at δ* over N if for any $\alpha \leq \delta$ and for N-generic $g \subseteq Coll(\omega, \alpha)$,

$$A \cap N[g] = p[T]^{N[g]} = \mathbb{R}^{N[g]} \backslash p[U]^{N[g]}.$$

We also say that N locally Suslin captures A at δ. We say that N locally captures A if N locally captures A at any uncountable cardinal of N. We say (N, Σ) *Suslin captures A at δ*, or (N, δ, Σ) *Suslin captures A*, if there are trees $T, U \in N$ on $\omega \times (\delta^+)^N$ such that whenever $i : N \to M$ comes from an iteration via Σ, $(i(T), i(U))$ locally Suslin captures A over M at $i(\delta)$. In this case we also say that $(N, \delta, \Sigma, T, U)$ Suslin captures A. We say (N, Σ) Suslin captures A if for every countable δ which is an uncountable cardinal of N, (N, Σ) Suslin captures A at δ. When δ is Woodin in N, one can perform genericity iterations on N to make various objects generic over an iterate of N. This is where the concept of Suslin capturing becomes interesting and useful. We exploit this fact on several occasions.

We say that Γ is a *good pointclass* if it is closed under recursive preimages, closed under $\exists^{\mathbb{R}}$, is ω-parametrized, and has the scale property. Furthermore, if Γ is closed under $\forall^{\mathbb{R}}$, then we say that Γ is *inductive-like*.

We quote a couple of theorems of Woodin, which will be key in our HOD analysis.

Theorem 2.3 (Woodin, see [2])**.** *Assume AD^+. Let $\langle \theta_\alpha \mid \alpha \leq \Omega \rangle$ be the Solovay sequence. Suppose $\alpha = 0$ or $\alpha = \beta + 1$ for some $\beta < \Omega$. Then $HOD \vDash \theta_\alpha$ is Woodin.*

Theorem 2.4 (Woodin)**.** *Assume $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$. Then $HOD = L[P]$ for some $P \subseteq \Theta$ in HOD.*

A proof of Theorem 2.4 can be found in [16, Theorem 3.1.9]. Next, we summarize some definitions and facts about hod mice that will be used in our computation. For basic definitions and notations that we omit, see [3]. The formal definition of a hod premouse $\mathcal{P}$ is given in Definition 2.12 of [3]. Let us mention some basic first-order properties of $\mathcal{P}$. There are an ordinal $\lambda^{\mathcal{P}}$ and sequences $\langle(\mathcal{P}(\alpha), \Sigma_\alpha^{\mathcal{P}}) \mid \alpha < \lambda^{\mathcal{P}}\rangle$ and $\langle\delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ such that

1. $\langle\delta_\alpha^{\mathcal{P}} \mid \alpha \leq \lambda^{\mathcal{P}}\rangle$ is increasing and continuous and if α is a successor ordinal then $\mathcal{P} \vDash \delta_\alpha^{\mathcal{P}}$ is Woodin;
2. $\mathcal{P}(0) = \mathrm{Lp}_\omega(\mathcal{P}|\delta_0)^{\mathcal{P}}$; for $\alpha < \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha+1) = (\mathrm{Lp}_\omega^{\Sigma_\alpha^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$; for limit $\alpha \leq \lambda^{\mathcal{P}}$, $\mathcal{P}(\alpha) = (\mathrm{Lp}_\omega^{\oplus_{\beta<\alpha}\Sigma_\beta^{\mathcal{P}}}(\mathcal{P}|\delta_\alpha))^{\mathcal{P}}$;
3. $\mathcal{P} \vDash \Sigma_\alpha^{\mathcal{P}}$ is a $(\omega, o(\mathcal{P}), o(\mathcal{P}))$[5]-strategy for $\mathcal{P}(\alpha)$ with hull condensation;
4. if $\alpha < \beta < \lambda^{\mathcal{P}}$ then $\Sigma_\beta^{\mathcal{P}}$ extends $\Sigma_\alpha^{\mathcal{P}}$.

We will write $\delta^{\mathcal{P}}$ for $\delta_{\lambda^{\mathcal{P}}}^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}} = \oplus_{\beta<\lambda^{\mathcal{P}}}\Sigma_{\beta+1}^{\mathcal{P}}$.

Definition 2.5. $(\mathcal{P}, \Sigma)$ *is a hod pair if* $\mathcal{P}$ *is a countable hod premouse and* Σ *is a* $(\omega, \omega_1, \omega_1)$ *iteration strategy for* $\mathcal{P}$ *with hull condensation such that* $\Sigma^{\mathcal{P}} \subseteq \Sigma$ *and this fact is preserved by* Σ*-iterations.*

Hod pairs typically arise in AD^+-models, where ω_1-iterability implies $\omega_1 + 1$-iterability. In practice, we work with hod pairs $(\mathcal{P}, \Sigma)$ such that Σ also has branch condensation. It follows from [3] that Σ is pullback consistent, positional, and commuting. Such hod pairs are particularly important for our computation as they are points in the direct limit system giving rise to HOD. For hod pairs $(\mathcal{M}_\Sigma, \Sigma)$, if Σ is a strategy with branch condensation and $\vec{\mathcal{T}}$ is a stack on $\mathcal{M}_\Sigma$ with last model $\mathcal{N}$, $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ is independent of $\vec{\mathcal{T}}$. Therefore, later on we will omit the subscript $\vec{\mathcal{T}}$ from $\Sigma_{\mathcal{N},\vec{\mathcal{T}}}$ whenever Σ is a strategy with branch condensation and $\mathcal{M}_\Sigma$ is a hod mouse.

Definition 2.6. *Suppose* $\mathcal{P}$ *and* $\mathcal{Q}$ *are two hod premice. Then* $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ *if there is* $\alpha \leq \lambda^{\mathcal{Q}}$ *such that* $\mathcal{P} = \mathcal{Q}(\alpha)$*.*

If $\mathcal{P}$ and $\mathcal{Q}$ are hod premice such that $\mathcal{P} \trianglelefteq_{hod} \mathcal{Q}$ then we say $\mathcal{P}$ is a hod initial segment of $\mathcal{Q}$. If $(\mathcal{P}, \Sigma)$ is a hod pair, and $\mathcal{Q} \trianglelefteq_{hod} \mathcal{P}$, say $\mathcal{Q} = \mathcal{P}(\alpha)$, then we let $\Sigma_{\mathcal{Q}}$ be the strategy of $\mathcal{Q}$ given by Σ. Note that $\Sigma_{\mathcal{Q}} \cap \mathcal{P} = \Sigma_\alpha^{\mathcal{P}} \in \mathcal{P}$.

All hod pairs $(\mathcal{P}, \Sigma)$ have the property that Σ has hull condensation and therefore, mice relative to Σ (or Σ-mice) make sense. To state the Strong Mouse Capturing we need to introduce the notion of Γ-fullness preservation. We fix some reasonable coding (we call Code) of $(\omega, \omega_1, \omega_1)$-strategies by sets of reals. Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. Let $I(\mathcal{P}, \Sigma)$ be

[5] This just means $\Sigma_\alpha^{\mathcal{P}}$ acts on all stacks of ω-maximal, normal trees in $\mathcal{P}$.

the set $(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \vec{\mathcal{T}})$ such that $\vec{\mathcal{T}}$ is according to Σ such that $i^{\vec{\mathcal{T}}}$ exists and $\mathcal{Q}$ is the end model of $\vec{\mathcal{T}}$ and $\Sigma_{\mathcal{Q}}$ is the $\vec{\mathcal{T}}$-tail of Σ. Let $B(\mathcal{P}, \Sigma)$ be the set $(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \vec{\mathcal{T}})$ such that there is some $\mathcal{R}$ such that $\mathcal{Q} = \mathcal{R}(\alpha)$, $\Sigma_{\mathcal{Q}} = \Sigma_{\mathcal{R}(\alpha)}$ for some $\alpha < \lambda^{\mathcal{R}}$ and $(\mathcal{R}, \Sigma_{\mathcal{R}}, \vec{\mathcal{T}}) \in I(\mathcal{P}, \Sigma)$.

Definition 2.7. *Suppose Σ is an iteration strategy with hull-condensation, a is a countable transitive set such that $\mathcal{M}_\Sigma \in a$[6] and Γ is a pointclass closed under boolean operations and continuous images and preimages. Then $Lp^{\Gamma,\Sigma}_{\omega_1}(a) = \bigcup_{\alpha<\omega_1} Lp^{\Gamma,\Sigma}_{\alpha}(a)$ where*

1. $Lp^{\Gamma,\Sigma}_0(a) = a \cup \{a\}$
2. $Lp^{\Gamma,\Sigma}_{\alpha+1}(a) = \cup\{\mathcal{M} : \mathcal{M}$ *is a sound Σ-mouse over* $Lp^{\Gamma,\Sigma}_{\alpha}(a)$[7] *projecting to* $Lp^{\Gamma,\Sigma}_{\alpha}(a)$ *and having an iteration strategy in* $\Gamma\}$.
3. $Lp^{\Gamma,\Sigma}_{\lambda}(a) = \bigcup_{\alpha<\lambda} Lp^{\Gamma,\Sigma}_{\alpha}(a)$ *for limit* λ.

We let $Lp^{\Gamma,\Sigma}(a) = Lp^{\Gamma,\Sigma}_1(a)$.

Definition 2.8 (Γ-Fullness preservation)**.** *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair and Γ is a pointclass closed under boolean operations and continuous images and preimages. Then Σ is a Γ-fullness preserving if whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha + 1 \leq \lambda^{\mathcal{Q}}$ and $\eta > \delta_\alpha$ is a strong cutpoint of $\mathcal{Q}(\alpha+1)$, then*

$$\mathcal{Q}|(\eta^+)^{\mathcal{Q}(\alpha+1)} = Lp^{\Gamma,\Sigma_{\mathcal{Q}(\alpha),\vec{\mathcal{T}}}}(\mathcal{Q}|\eta).$$

and

$$\mathcal{Q}|(\delta_\alpha^+)^{\mathcal{Q}} = Lp^{\Gamma,\oplus_{\beta<\alpha}\Sigma_{\mathcal{Q}(\beta+1),\vec{\mathcal{T}}}}(\mathcal{Q}|\delta_\alpha^{\mathcal{Q}}).$$

When $\Gamma = \wp(\mathbb{R})$, we simply say fullness preservation; in this case, we also write Lp (Lp^Σ) instead of Lp^Γ ($\mathrm{Lp}^{\Gamma,\Sigma}$). A stronger notion of Γ-fullness preservation is super Γ-fullness preservation. Similarly, when $\Gamma = \wp(\mathbb{R})$, we simply say super fullness preservation.

Definition 2.9 (Super Γ-fullness preserving)**.** *Suppose $(\mathcal{P}, \Sigma)$ is a hod pair and Γ is a pointclass closed under boolean operations and continuous images and preimages. Σ is super Γ-fullness preserving if it is Γ-fullness preserving and whenever $(\vec{\mathcal{T}}, \mathcal{Q}) \in I(\mathcal{P}, \Sigma)$, $\alpha < \lambda^{\mathcal{Q}}$ and $x \in HC$ is generic over $\mathcal{Q}$, then*

$$Lp^{\Gamma,\Sigma_{\mathcal{Q}(\alpha)}}(x) = \{\mathcal{M} \mid \mathcal{Q}[x] \models \text{“}\mathcal{M} \textit{ is a sound } \Sigma_{\mathcal{Q}(\alpha)}\textit{-mouse over } x \textit{ and } \rho_\omega(\mathcal{M}) = x\text{”}\}.$$

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[6] $\mathcal{M}_\Sigma$ is the structure that Σ-iterates.

[7] By this we mean $\mathcal{M}$ has a unique (ω, ω_1+1)-iteration strategy Λ above $\mathrm{Lp}^{\Gamma,\Sigma}_{\alpha}(a)$ such that whenever $\mathcal{N}$ is a Λ-iterate of $\mathcal{M}$, then $\mathcal{N}$ is a Σ-premouse.

Moreover, for such an $\mathcal{M}$ as above, letting Λ be the unique strategy for $\mathcal{M}$, then for any cardinal κ of $\mathcal{Q}[x]$, $\Lambda \restriction H_\kappa^{\mathcal{Q}[x]} \in \mathcal{Q}[x]$.

By [3], hod pairs $(\mathcal{P}, \Sigma)$ such that Σ is super fullness preserving exist under $(*)$. These pairs typically come from hod pair (or Γ-hod pair) constructions (see [3, Lemma 3.2.3]). Hod mice that go into the direct limit system that gives rise to HOD have strategies that are super fullness preserving and have branch condensation. Here is the statement of the strong mouse capturing.

Definition 2.10 (The Strong Mouse Capturing)**.** *The Strong Mouse Capturing (SMC) is the statement: Suppose $(\mathcal{P}, \Sigma)$ is a hod pair such that Σ has branch condensation and is Γ-fullness preserving for some Γ. Then for any $x, y \in \mathbb{R}$, $x \in OD_\Sigma(y)$ if and only if x is in some Σ-mouse over $\langle \mathcal{P}, y \rangle$.*

When $(\mathcal{P}, \Sigma) = \emptyset$ in the statement of Definition 2.10 we get the ordinary Mouse Capturing (MC). The Strong Mouse Set Conjecture (SMSC) just conjectures that SMC holds below a superstrong.

Definition 2.11 (Strong Mouse Set Conjecture)**.** *Assume AD^+ and that there is no mouse with a superstrong cardinal. Then SMC holds.*

Recall that by results of [3], SMSC holds assuming $(*)$. To prove that hod pairs exist in AD^+ models, we typically do a hod pair construction (or a Γ-hod pair construction for some pointclass Γ). For the details of these constructions, see Definitions 2.1.8 and 2.2.5 in [3].

Suppose Γ is a pointclass closed under complements and under continuous preimages. Suppose also that $\lambda^{\mathcal{P}}$ is limit. We let

$$\Gamma(\mathcal{P}, \Sigma) = \{A \mid \exists(\mathcal{Q}, \Sigma_{\mathcal{Q}}, \vec{\mathcal{T}}) \in B(\mathcal{P}, \Sigma)\ A <_w {}^{8} Code(\Sigma_{\mathcal{Q}})\}.$$

$$HP^\Gamma = \{(\mathcal{P}, \Lambda) \mid (\mathcal{P}, \Lambda) \text{ is a hod pair and } Code(\Lambda) \in \Gamma\},$$

and

$$\begin{aligned} Mice^\Gamma = \{(a, \Lambda, \mathcal{M}) \quad & \mid a \in HC,\ a \text{ is self-wellordered transitive, } \Lambda \text{ is an iteration} \\ & \text{strategy such that } (\mathcal{M}_\Lambda, \Lambda) \in HP^\Gamma,\ \mathcal{M}_\Lambda \in a, \text{ and } \mathcal{M} \trianglelefteq \mathrm{Lp}^{\Gamma,\Lambda}(a)\}. \end{aligned}$$

If $\Gamma = \wp(\mathbb{R})$, we let $HP = HP^\Gamma$ and $Mice = Mice^\Gamma$. Suppose $(\mathcal{M}_\Sigma, \Sigma) \in HP^\Gamma$. Let

$$Mice^\Gamma_\Sigma = \{(a, \mathcal{M}) \mid (a, \Sigma, \mathcal{M}) \in Mice^\Gamma\}.$$

[8] Wadge reducible to

2.2 A Prikry forcing

Let $(\mathcal{P}, \Sigma)$ be a hod pair such that Σ has branch condensation and $\mathrm{Lp}^{\Sigma}(\mathbb{R})$ is defined and satisfies AD^{+} (see [7] for a precise definition of $\mathrm{Lp}^{\Sigma}(\mathbb{R})$). We briefly describe a notion of Prikry forcing that will be useful in our HOD computation. The forcing $\mathbb{P}$ described here is defined in $\mathrm{Lp}^{\Sigma}(\mathbb{R})$ and is a modification of the forcing defined in Section 6.6 of [9] or in [5]. All facts about this forcing are proved similarly as those in Section 6.6 of [9] so we omit all proofs.

First, let T be the tree of a $\Sigma^2_1(\Sigma)$ scale on a universal $\Sigma^2_1(\Sigma)$ set U. Write $\mathcal{P}_x$ for the Σ-premouse[9] coded by the real x. Let a be countable transitive, $x \in \mathbb{R}$ such that a is coded by a real recursive in x. A normal iteration tree $\mathcal{U}$ on a 0-suitable Σ-premouse $\mathcal{Q}$ (see [5] or Definition 4.1, where $(\mathcal{Q}, \Sigma)$ is defined to be 0-suitable) is *short* if for all limit $\xi \leq lh(\mathcal{U})$, $\mathrm{Lp}^{\Sigma}(\mathcal{M}(\mathcal{U}|\xi)) \vDash \delta(\mathcal{U}|\xi)$ is not Woodin. Otherwise, we say that $\mathcal{U}$ is *maximal*. We say that a 0-suitable $\mathcal{P}_z$ is short-tree iterable by Λ if for any short tree $\mathcal{T}$ on $\mathcal{P}_z$, $b = \Lambda(\mathcal{T})$ is such that $\mathcal{M}^{\mathcal{T}}_b$ is 0-suitable, and b has a Q-structure $\mathcal{Q}$ such that $\mathcal{Q} \trianglelefteq \mathcal{M}^{\mathcal{T}}_b$. Put

$$\mathcal{F}^x_a = \{\mathcal{P}_z \mid z \leq_T x, \mathcal{P}_z \text{ is a short-tree iterable 0-suitable } \Sigma\text{-premouse over } a\}$$

For each a, for x in the cone in the previous claim, working in $L[T, x]$, we can simultaneously compare all $\mathcal{P}_z \in \mathcal{F}^x_a$ (using their short-tree iteration strategy) while doing the genericity iterations to make all y such that $y \leq_T x$ generic over the common part of the final model $\mathcal{Q}^{x,-}_a$. This process (hence $\mathcal{Q}^{x,-}_a$) depends only on the Turing degree of x. Put

$$\mathcal{Q}^x_a = \mathrm{Lp}^{\Sigma}_{\omega}(\mathcal{Q}^{x,-}_a), \text{ and } \delta^x_a = o(\mathcal{Q}^{x,-}_a).$$

By the above discussion, $\mathcal{Q}^x_a, \delta^x_a$ depend only on the Turing degree of x. Here are some properties obtained from the above process.

1. $\mathcal{F}^x_a \neq \emptyset$ for x of sufficiently large degree;
2. $\mathcal{Q}^{x,-}_a$ is full (no levels of $\mathcal{Q}^x_a$ project strictly below δ^x_a);
3. $\mathcal{Q}^x_a \vDash \delta^x_a$ is Woodin;
4. $\wp(a) \cap \mathcal{Q}^x_a = \wp(a) \cap OD_T(a \cup \{a\})$ and $\wp(\delta^x_a) \cap \mathcal{Q}^x_a = \wp(\delta^x_a) \cap OD_T(Q^{x,-}_a \cup \{Q^{x,-}_a\})$;
5. $\delta^x_a = \omega_1^{L[T,x]}$.

Now for an increasing sequence $\vec{d} = \langle d_0, ..., d_n \rangle$ of Turing degrees, and a countable transitive, set

[9] To be entirely correct, we should call $\mathcal{P}_x$ a reorganized Σ-premouse in the sense of [3, 2.10], but we suppress this cumbersome name here.

$$\mathcal{Q}_0(a) = \mathcal{Q}_a^{d_0} \text{ and } \mathcal{Q}_{i+1}(a) = \mathcal{Q}_{\mathcal{Q}_i(a)}^{d_{i+1}} \text{ for } i < n$$

We assume from here on that the degrees d_{i+1}'s are such that $\mathcal{Q}_{\mathcal{Q}_i(a)}^{d_{i+1}}$ are defined. For $\vec{d}$ as above, write $\mathcal{Q}_i^{\vec{d}}(a) = \mathcal{Q}_i(a)$ even though $\mathcal{Q}_i(a)$ only depends on $\vec{d}|(i+1)$. Let μ be the cone measure on the Turing degrees. We can then define our Prikry forcing $\mathbb{P}$ (over $L(T, \mathbb{R})$) as follows. A condition $(p, S) \in \mathbb{P}$ just in case $p = \langle \mathcal{Q}_0^{\vec{d}}(a), ..., \mathcal{Q}_n^{\vec{d}}(a) \rangle$ for some $\vec{d}$, $S \in L(T, \mathbb{R})$ is a "measure-one tree" consisting of stems q which either are initial segments or end-extensions of p and such that $(\forall q = \langle \mathcal{Q}_0^{\vec{e}}(a), ..., \mathcal{Q}_k^{\vec{e}}(a) \rangle \in S)(\forall_\mu^* d)$ let $\vec{f} = \langle \vec{e}(0), ..., \vec{e}(k), d \rangle$, we have $\langle \mathcal{Q}_0^{\vec{f}}(a), ..., \mathcal{Q}_{(k+1)}^{\vec{f}}(a) \rangle \in S$. The ordering on $\mathbb{P}$ is defined as follows.

$$(p, S) \preccurlyeq (q, W) \Leftrightarrow p \text{ end-extends } q,\ S \subseteq W, \text{ and } \forall n \in \operatorname{dom}(p) \setminus \operatorname{dom}(q)\ (p|(n+1) \in W).$$

$\mathbb{P}$ has the Prikry property, that is, for any formula $\varphi(v)$ in the forcing language, for any $\mathbb{P}$-term τ in $\mathrm{Lp}^\Sigma(\mathbb{R})$, for any condition $(p, S) \in \mathbb{P}$, there is a condition $(p, S^*) \preccurlyeq (p, S)$ such that either $(p, S^*) \Vdash \varphi[\tau]$ or $(p, S^*) \Vdash \neg\varphi[\tau]$. Let G be a $\mathbb{P}$-generic over $\mathrm{Lp}^\Sigma(\mathbb{R})$, $\langle \mathcal{Q}_i \mid i < \omega \rangle = \cup\{p \mid \exists \vec{X}(p, \vec{X}) \in G\}$ and $\mathcal{Q}_\infty = \bigcup_i \mathcal{Q}_i$. Let δ_i be the largest Woodin cardinal of $\mathcal{Q}_i$. Then

$$P(\delta_i) \cap L[T, \langle \mathcal{Q}_i \mid i < \omega \rangle] \subseteq \mathcal{Q}_i,$$

and

$$L[T, \mathcal{Q}_\infty] = L[T, \langle \mathcal{Q}_i \mid i < \omega \rangle] \vDash \delta_i \text{ is Woodin.}$$

Definition 2.12 (Derived models)**.** *Suppose $M \vDash \mathsf{ZFC}$ and $\lambda \in M$ is a limit of Woodin cardinals in M. Let $G \subseteq Col(\omega, <\lambda)$ be generic over M. Let $\mathbb{R}_G^*$ (or just $\mathbb{R}^*$) be the symmetric reals of $M[G]$ and Hom_G^* (or just Hom^*) be the set of $A \subseteq \mathbb{R}^*$ in $M(\mathbb{R}^*)$ such that there is a tree T such that $A = p[T] \cap \mathbb{R}^*$ and there is some $\alpha < \lambda$ such that*

$$M[G \restriction \alpha] \vDash \text{“}T \textit{ has a } {<}\text{-}\lambda\textit{-complement”}.$$

By the ***old derived model*** *of M at λ, denoted by $D(M, \lambda)$, we mean the model $L(\mathbb{R}^*, Hom^*)$. By the* ***new derived model*** *of M at λ, denoted by $D^+(M, \lambda)$, we mean the model $L(\Gamma, \mathbb{R}^*)$, where Γ is the closure under Wadge reducibility of the set of $A \in M(\mathbb{R}^*) \cap \wp(\mathbb{R}^*)$ such that $L(A, \mathbb{R}^*) \vDash \mathsf{AD}^+$.*

Theorem 2.13 (Woodin)**.** *Let M be a model of ZFC and $\lambda \in M$ be a limit of Woodin cardinals of M. Then $D(M, \lambda) \vDash \mathsf{AD}^+$, $D^+(M, \lambda) \vDash \mathsf{AD}^+$. Furthermore, Hom^* is the pointclass of Suslin co-Suslin sets of $D^+(M, \lambda)$.*

Using the proof of Theorem 0.1 from [5], we get that (definably over) $\mathrm{Lp}^{\Sigma}(\mathbb{R})[G]$, there is a Σ-premouse $\mathcal{Q}^{+}_{\infty}$ extending $\mathcal{Q}_{\infty}$ such that $\mathrm{Lp}^{\Sigma}(\mathbb{R})$ can be realized as a (new) derived model of $\mathcal{Q}^{+}_{\infty}$ at ω_1^V, which is the limit of Woodin cardinals of $\mathcal{Q}^{+}_{\infty}$. The Σ-premouse $\mathcal{Q}^{+}_{\infty}$ is the union of Σ-premice $\mathcal{R}$ over $\mathcal{Q}_{\infty}$, where $\mathcal{R}$ is an S-translation of some $\mathcal{M} \lhd \mathrm{Lp}^{\Sigma}(\mathbb{R})$ (see [3] for more on S-translations).

Finally, we remark that Theorem 0.1 of [5] shows that if $(*)$ holds and in addition, $\Theta = \theta_0$ or $\Theta = \theta_{\alpha+1}$ for some α, then there is a hod pair $(\mathcal{P}, \Sigma)$ ($(\mathcal{P}, \Sigma) = (\emptyset, \emptyset)$ if $\Theta = \theta_0$) such that $V = L(\mathrm{Lp}^{\Sigma}(\mathbb{R}))$[10]. We write $\mathrm{Lp}(\mathbb{R})$ if $\Sigma = \emptyset$. This particular representation of V and the discussion above will be useful for the HOD computation in Section 4.

3 The Limit Case

There are two cases: the easier case is when $\mathrm{HOD} \vDash$ "$\mathrm{cof}(\Theta)$ is not measurable", and the harder case is when $\mathrm{HOD} \vDash$ "$\mathrm{cof}(\Theta)$ is measurable".

Here's the direct limit system that gives us $V_{\Theta}^{\mathrm{HOD}}$.

$$\mathcal{F} = \{(\mathcal{Q}, \Lambda) \mid (\mathcal{Q}, \Lambda) \text{ is a hod pair; } \Lambda \text{ is fullness preserving and has branch condensation}\}.$$

The order on $\mathcal{F}$ is given by

$$(\mathcal{Q}, \Lambda) \leq^{\mathcal{F}} (\mathcal{R}, \Psi) \iff \mathcal{Q} \text{ iterates to a hod initial segment of } \mathcal{R}.$$

$\leq^{\mathcal{F}}$ is directed and we can form the direct limit of $\mathcal{F}$ under the natural embeddings coming from the comparison process. Let $\mathcal{M}_{\infty}$ be the direct limit. By the computation in [3],

$$|\mathcal{M}_{\infty}| = V_{\Theta}^{\mathrm{HOD}}.$$

$\mathcal{M}_{\infty}$ as a structure also has a predicate for its extender sequence and a predicate for a sequence of strategies.

Proof of Theorem 1.1. To prove the theorem, suppose the equality is false. Then by Theorem 2.4, there is an $A \subseteq \Theta$ such that $A \in \mathrm{HOD} \backslash L(\mathcal{M}_{\infty})$ (the fact that $L(\mathcal{M}_{\infty}) \subseteq \mathrm{HOD}$ follows from the definition of $\mathcal{M}_{\infty}$). By Σ_1-reflection ([8, Theorem 1]), there is a transitive N coded

[10] By [3], Σ is in fact a (Θ, Θ)-iteration strategy and $\mathcal{M}_1^{\Sigma,\sharp}$ is (Θ, Θ)-iterable. This is enough iterability to define $\mathrm{Lp}^{\Sigma}(\mathbb{R})$ as in [7].

by a Suslin co-Suslin set such that

$$N \models \mathsf{ZF}^- + \mathsf{AD}^+ + V = L(\wp(\mathbb{R})) + \mathsf{SMC} + \Theta \text{ exists and is limit in the Solovay sequence}$$
$$+ \text{HOD} \models \text{“cof}(\Theta) \text{ is not measurable ”} + \text{“}\exists B \subseteq \Theta(B \in \text{HOD}\backslash L(\mathcal{M}_\infty))\text{”}.$$

Take N to be the minimal such and let B witness the failure of the theorem in N. Let $\Omega = \wp(\mathbb{R})^N$. Fix a coarsely-iterable mouse N_x^* (in V) capturing a good pointclass Ω' beyond Ω, i.e. $\Omega \subsetneq \mathbf{\Delta}_{\Omega'}$ (see [3] or [13, Theorem 10.3] for more on N_x^*, where x is a real on some cone where the operation $x \mapsto N_x^*$ is defined). Let ϕ define B (for simplicity, we suppress the ordinal parameter) i.e.

$$\alpha \in B \;\Leftrightarrow\; N \models \phi[\alpha].$$

There is a pair $(\mathcal{P}, \Sigma)$, a limit ordinal λ, a sequence $(\mathcal{P}_\gamma, \Sigma_\gamma \mid \gamma < \lambda)$, such that:

1. $(\bigcup_{\gamma<\lambda} \mathcal{P}_\gamma) \lhd \mathcal{P}$;
2. for all $\beta < \lambda$, $\mathcal{P}_\beta$ is the β-th hod mouse in the Ω-hod pair construction of N_x^* (so $\lambda^{\mathcal{P}_\beta} = \beta$). So if $\gamma < \eta < \lambda$, $\mathcal{P}_\gamma \trianglelefteq_{hod} \mathcal{P}_\eta$;
3. $\Sigma_\gamma \in \Omega$ is the corresponding strategy of $\mathcal{P}_\gamma$ which is Ω-fullness preserving, has branch condensation;
4. $\mathcal{P}$ is the first model $\mathcal{Q}$ of the $L[E, \oplus_{\gamma<\lambda}\Sigma_\gamma][\bigcup_{\gamma<\lambda} \mathcal{P}_\gamma]$-construction of N_x^* such that
$$\rho_\omega(\mathcal{Q}) < o(\textstyle\bigcup_{\gamma<\lambda^{\mathcal{P}}} \mathcal{P}_\gamma));$$
5. Σ is the induced strategy for $\mathcal{P}$ with branch condensation and extends $\oplus_{\gamma<\lambda^{\mathcal{P}}}\Sigma_\gamma$;

That λ is limit and $\rho_\omega(\mathcal{P}) < o(\bigcup_{\gamma<\lambda^{\mathcal{P}}} \mathcal{P}_\gamma)$ are because of the choice of N_x^*, Ω' and Θ being limit in the Solovay sequence.

Let $(\delta_\beta^{\mathcal{P}} \mid \beta < \lambda)$ be the Woodin cardinals of $\mathcal{P}$ below $o(\bigcup_{\gamma<\lambda^{\mathcal{P}}} \mathcal{P}_\gamma))$. Let $\lambda^{\mathcal{P}} = \lambda$ and $\delta^{\mathcal{P}} = \sup_{\gamma<\lambda}\delta_\gamma^{\mathcal{P}}$. We also use similar notations for Σ-iterates of $\mathcal{P}$. For a Σ-iterate $\mathcal{R}$ of $\mathcal{P}$, we let $\Sigma_{\mathcal{R}}$ be the tail of Σ. By going to a Σ-iterate if necessary, we assume that $(\bigcup_{\gamma<\lambda^{\mathcal{P}}} \mathcal{P}_\gamma, \oplus_{\gamma<\lambda^{\mathcal{P}}}\Sigma_\gamma)$ satisfies [3, Theorem 3.2.21] applied in N. This and the choice of $\mathcal{P}$ imply that the direct limit $\mathcal{M}_\infty^+$ of all Σ-iterates of $\mathcal{P}$[11] extends $(\mathcal{M}_\infty)^N$ and $(\Sigma_\gamma \mid \gamma < \lambda)$ is cofinal in Ω. Hence, letting $j : \mathcal{P} \to \mathcal{M}_\infty^+$ be the natural map, then $\mathcal{M}_\infty^+ | j(\delta^{\mathcal{P}}) = (\mathcal{M}_\infty)^N$.

Now pick a sequence $\langle \gamma_i \mid i < \omega \rangle$ cofinal in $\lambda^{\mathcal{P}}$ such that $\delta_{\lambda^{\mathcal{P}_{\gamma_i}}}$ is Woodin in $\mathcal{P}$, an enumeration $\langle x_i \mid i < \omega \rangle$ of $\mathbb{R}$ and do a genericity iteration of $\mathcal{P}$ to successively make each x_i

[11] $\mathcal{P}$ is what is called an “anomalous hod mouse” in [3]. See [3, Section 2.7] for how to iterate anomalous hod mice.

generic at appropriate image of $\delta_{\lambda^{\mathcal{P}_{\gamma_i}}}$. Let $\mathcal{Q}$ be the end model of this process and $i : \mathcal{P} \to \mathcal{Q}$ be the iteration embedding.[12] We have that N is the derived model of $\mathcal{Q}$ at $i(\delta^{\mathcal{P}})$ and in fact, $\mathcal{P} = L_\beta(\bigcup_{\gamma<\lambda} \mathcal{P}_\gamma)$ for some β.

Let D be the derived model of $\mathcal{M}_\infty^+$ at Θ and

$$\pi_\infty : \mathcal{M}_\infty \to (\mathcal{M}_\infty)^D$$

be the direct limit embedding given by the join of the strategies of $\mathcal{M}_\infty$'s hod initial segments (this is the tail of $\oplus_{\gamma<\lambda}\Sigma_\gamma$). We claim that

$$\alpha \in B \;\Leftrightarrow\; D \vDash \phi[\pi_\infty(\alpha)] \qquad (\dagger).$$

To see (†), suppose not and let α be a counter-example. So

$$N \vDash \phi[\alpha] \Leftrightarrow D \vDash \neg\phi[\pi_\infty(\alpha)].$$

Let $\mathcal{R}$ be a Σ-iterate of $\mathcal{P}$ ($\mathcal{R}$ is countable) such that letting $\pi_{\mathcal{R},\infty} : \mathcal{R} \to \mathcal{M}_\infty^+$ be the direct limit map, there is some $\alpha^* \in \mathcal{R}$ such that $\pi_{\mathcal{R},\infty}(\alpha^*) = \alpha$. We may assume $\mathcal{Q}$ is an iterate of $\mathcal{R}$ and let $\pi_{\mathcal{R},\mathcal{Q}}$ be the iteration map. Let

$$\pi_{\mathcal{Q},\infty} : \mathcal{Q}|\delta^{\mathcal{Q}} \to (\mathcal{M}_\infty)^{D(\mathcal{Q},\delta^{\mathcal{Q}})} = (\mathcal{M}_\infty)^N$$

be the direct limit map. Note that $\Sigma_{\mathcal{Q}}^{\mathcal{Q}} \restriction (\pi_{\mathcal{R},\mathcal{Q}}(\alpha^*) + 1) \in \mathcal{Q}$ determines the map $\pi_{\mathcal{Q},\infty} \restriction (\pi_{\mathcal{R},\mathcal{Q}}(\alpha^*) + 1)$. So the statement "$N \vDash \phi[\alpha]$" is definable over $\mathcal{Q}$ via a formula φ from parameters $\Sigma_{\mathcal{Q}}^{\mathcal{Q}} \restriction (\pi_{\mathcal{R},\mathcal{Q}}(\alpha^*) + 1)$ and $\pi_{\mathcal{R},\mathcal{Q}}(\alpha^*)$, i.e.

$$N \vDash \phi[\alpha] \Leftrightarrow \mathcal{Q} \vDash \varphi[\Sigma_{\mathcal{Q}}^{\mathcal{Q}} \restriction (\pi_{\mathcal{R},\mathcal{Q}}(\alpha^*) + 1), \pi_{\mathcal{R},\mathcal{Q}}(\alpha^*)].$$

By elementarity,

$$\mathcal{M}_\infty^+ \vDash \varphi[\Sigma_{\mathcal{M}_\infty^+}^{\mathcal{M}_\infty^+} \restriction (\alpha + 1), \alpha].$$

But this means $D \vDash \phi[\pi_\infty(\alpha)]$. Contradiction.

(†) implies that $B \in \mathcal{M}_\infty^+ = (L[\mathcal{M}_\infty])^N$, which contradicts our assumption. Hence we're done. □

Remark 3.1. *It's not clear that in the statement of Theorem 1.1, "$\mathcal{M}_\infty$" can be replaced by "V_Θ^{HOD}".*

[12] We actually don't know that Σ can be extended to $V^{\mathrm{Col}(\omega,\mathbb{R})}$ but we can replace N by a countable model. See [9, page 72] for the detail. From now on, we pretend that Σ is iterable in $V^{\mathrm{Col}(\omega,\mathbb{R})}$ without further comment.

Suppose HOD $\vDash$ cof(Θ) is measurable. The computation in this case is more complicated due to the fact that iterations of a hod mouse $\mathcal{P}$ such that $\mathcal{P} \vDash$ "cof($\lambda^{\mathcal{P}}$) is measurable" may introduce new Woodin cardinals in the iterates above the pointwise images of $\lambda^{\mathcal{P}}$.

Recall that by [3], V_Θ^{HOD} is the universe of $\mathcal{M}_\infty$ where $\mathcal{M}_\infty$ is the direct limit (under the natural maps) of the directed system $\mathcal{F}$. Let

$$\mathcal{M}_\infty^* = Ult_0(\mathrm{HOD}, \mu)|\Theta,$$

where μ is the order zero measure on $\mathrm{cof}^{\mathrm{HOD}}(\Theta)$. Let $f : \mathrm{cof}^{\mathrm{HOD}}(\Theta) =_{def} \alpha \to \Theta$ be a continuous and cofinal function in HOD. For each $\beta < \alpha$, let Λ_β be the strategy of $\mathcal{M}_\infty^*(f(\beta))$ and Σ_β be the strategy of $\mathcal{M}_\infty(f(\beta))$. Let

$$\mathcal{M}_\infty^+ = Ult_0(\mathrm{HOD}, \mu)|(\Theta^+)^{Ult_0(\mathrm{HOD},\mu)},$$

and

$$\mathcal{N}_\infty^+ = \bigcup\{\mathcal{M} \mid \mathcal{M}_\infty \trianglelefteq \mathcal{M},\ \rho(\mathcal{M}) = \Theta, \mathcal{M} \text{ is a hybrid mouse satisfying property } (\dagger)\}.$$

Here a mouse $\mathcal{M}$ satisfies property ($\dagger$) if whenever $\pi : \mathcal{M}^* \to \mathcal{M}$ is elementary, $\mathcal{M}^*$ is countable, transitive, and $\pi(\Theta^*) = \Theta$, then $\mathcal{M}^*$ is a $\oplus_{\xi<\Theta^*}\Sigma_\xi^*$-mouse for stacks above Θ^*, where Σ_ξ^* is the strategy for the hod mouse $\mathcal{M}^*(\xi)$ obtained by the following process: let $(\mathcal{P}, \Sigma) \in \mathcal{F}$ and $i : \mathcal{P} \to \mathcal{M}_\infty$ be the direct limit embedding such that the range of i contains the range of $\pi \restriction \mathcal{M}^*(\xi)$; Σ_ξ^* is then defined to be the $\pi \circ i^{-1}$-pullback of Σ. It's easy to see that the strategy Σ_ξ^* as defined doesn't depend on the choice of $(\mathcal{P}, \Sigma)$. This is because if $(\mathcal{P}_0, \Sigma_0, i_0)$ and $(\mathcal{P}_1, \Sigma_1, i_1)$ are two possible choices to define Σ_ξ^*, we can coiterate $(\mathcal{P}_0, \Sigma_0)$ against $(\mathcal{P}_1, \Sigma_1)$ to a pair $(\mathcal{R}, \Lambda)$ and let $i_i : \mathcal{P}_i \to \mathcal{R}$ be the iteration maps and let $i_2 : \mathcal{R} \to \mathcal{M}_\infty$ be the direct limit embedding. Then $\Sigma_0 = \Lambda^{i_0}$ and $\Sigma_1 = \Lambda^{i_1}$; hence the $\pi \circ i_0^{-1}$-pullback of Σ_0 is the same as the $\pi \circ i_1^{-1}$-pullback of Σ_1 because both are the same as the $\pi \circ i_2^{-1}$-pullback of Λ.

Proof of Theorem 1.2. To prove (1), first let $j_\mu : \mathrm{HOD} \to Ult_0(\mathrm{HOD}, \mu)$ be the canonical ultrapower map. Let $A \in \mathrm{HOD}$, $A \subseteq \Theta$. By the computation of HOD below Θ, we know that for each limit $\beta < \alpha$,

$$A \cap \theta_{f(\beta)} \in |\mathcal{M}_\infty(f(\beta))|.$$

This means

$$j_\mu(A) \cap \Theta \in \mathcal{M}_\infty^+.$$

We then have

$$\gamma \in A \Leftrightarrow j_\mu(\gamma) \in j_\mu(A) \cap \Theta.$$

Since $j_\mu|\Theta$ agrees with the canonical ultrapower map $k : \mathcal{M}_\infty \to Ult_0(\mathcal{M}_\infty, \mu)$ on all ordinals less than Θ, the above equivalence shows that $A \in L(\mathcal{M}_\infty, \mathcal{M}_\infty^+)$. This proves (1).

Suppose the statement of (2) is false. There is an $A \subseteq \Theta$ such that $A \in \mathrm{HOD}\backslash\mathcal{N}_\infty^+$. By Σ_1-reflection ([8, Theorem 1]), there is a transitive N coded by a Suslin co-Suslin set such that

$$\begin{aligned} N \models \ & \mathsf{ZF}^- + \mathsf{DC} + V = L(\wp(\mathbb{R})) + \mathsf{SMC} + \text{“}\Theta \text{ exists and is limit in the Solovay sequence ”} \\ & + \text{“HOD} \models \mathrm{cof}(\Theta) = \alpha \text{ is measurable as witnessed by } f\text{”} \\ & + \text{“}\exists A \subseteq \Theta (A \in \mathrm{HOD}\backslash\mathcal{N}_\infty^+)\text{”}. \end{aligned}$$

Take N to be the minimal such and let A witness the failure of (2) in N. Let μ, j_μ, $\mathcal{M}_\infty$, $\mathcal{M}_\infty^+$, $\mathcal{M}_\infty^*$, $\mathcal{N}_\infty^+$ be as above but relativized to N. Working in N, there is a sequence $\langle \mathcal{M}_\beta \mid \beta < \alpha, \beta \text{ is limit}\rangle \in \mathrm{HOD}$ such that for each limit $\beta < \alpha$, $\mathcal{M}_\beta$ is the least hod initial segment of $\mathcal{M}_\infty|\theta_{f(\beta)}$ such that $A \cap \theta_{f(\beta)}$ is definable over $\mathcal{M}_\beta$.

Let $\Omega = \wp(\mathbb{R})^N$. Fix an N_x^* capturing a good pointclass beyond Ω. Now, we again do the Ω-hod pair construction in N_x^* to obtain a pair $(\mathcal{Q}, \Lambda)$ such that

1. there is a limit ordinal $\lambda^\mathcal{Q}$ such that for all $\gamma < \lambda^\mathcal{Q}$, $\mathcal{Q}_\beta$ is a hod mouse with $\lambda^{\mathcal{Q}_\beta} = \beta$ and whose strategy $\Psi_\gamma \in \Omega$ is Ω-fullness preserving, has branch condensation;

2. if $\gamma < \eta < \lambda^\mathcal{Q}$, $\mathcal{Q}_\gamma \trianglelefteq_{hod} \mathcal{Q}_\eta$;

3. $\mathcal{Q}$ is the first sound mouse from the $L[E, \oplus_{\gamma<\lambda^\mathcal{Q}}\Psi_\gamma][\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma]$-construction done in N_x^* that has projectum $\leq o(\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma)$ and extends $\mathrm{Lp}^{\Omega,\oplus_{\gamma<\lambda^\mathcal{Q}}\Psi_\gamma}(\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma)$ [13] and Λ be the induced strategy of $\mathcal{Q}$.

From the construction of $\mathcal{Q}$ and the properties of N, it's easy to verify the following:

(a) Let $\delta_{\lambda^\mathcal{Q}} = o(\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma)$ and $\eta = o(\mathrm{Lp}^{\Omega,\oplus_{\gamma<\lambda^\mathcal{Q}}\Psi_\gamma}(\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma))$. Then $\eta = (\delta_{\lambda^\mathcal{Q}}^+)^\mathcal{Q}$.

(b) $\Lambda \notin \Omega$.

(c) $\mathcal{Q} \models \delta_{\lambda^\mathcal{Q}}$ has measurable cofinality.

Let $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the direct limit (under natural embeddings) of Λ-iterates of $\mathcal{Q}$.

[13] If $\mathcal{M} \lhd \mathrm{Lp}^{\Omega,\oplus_{\gamma<\lambda^\mathcal{Q}}\Psi_\gamma}(\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma)$ and $\mathcal{M}$ extends $\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma$ then $\mathcal{M}$ is a mouse in N in the sense that N knows how to iterate $\mathcal{M}$ for stacks above $o(\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma)$.

Lemma 3.2. $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ *exists.*

Proof. First note that Λ is Ω-fullness preserving. To see this, suppose not. Let $k : \mathcal{Q} \to \mathcal{R}$ be according to Λ witnessing this. It's easy to see that the tail $\Lambda_\mathcal{R}$ of Λ acting on $\mathcal{R}|k(\eta)$ is not in Ω (otherwise, $\Lambda^k_\mathcal{R} = \Lambda$ by hull condensation and hence $\Lambda \in \Omega$. Contradiction.) However, $\oplus_{\gamma<\lambda^\mathcal{R}} \Psi_{\mathcal{R}(\gamma)} \in \Omega$ since the iterate of N^*_x by the lift-up of k thinks that the fragment of its strategy inducing $\oplus_{\gamma<\lambda^\mathcal{R}} \Psi_{\mathcal{R}(\gamma)}$ is in Ω. Now suppose $\mathcal{M}$ is a $\oplus_{\gamma<\lambda^\mathcal{R}} \Psi_{\mathcal{R}(\gamma)}$-mouse projecting to $\delta_{\lambda^\mathcal{R}}$ with strategy Ξ in Ω and $\mathcal{M} \ntrianglelefteq \mathcal{R}$ (again, Ξ acts on trees above $\delta_{\lambda^\mathcal{R}}$ and moves the predicates for $\oplus_{\gamma<\lambda^\mathcal{R}} \Psi_{\mathcal{R}(\gamma)}$ correctly). We can compare $\mathcal{M}$ and $\mathcal{R}$ (the comparison is above $\delta_{\lambda^\mathcal{R}}$). Let $\overline{\mathcal{M}}$ be the last model on the $\mathcal{M}$ side and $\overline{\mathcal{R}}$ on the $\mathcal{R}$ side. Then $\overline{\mathcal{R}} \lhd \overline{\mathcal{M}}$. Let $\pi : \mathcal{R} \to \overline{\mathcal{R}}$ be the iteration map from the comparison process and Σ be the $\pi \circ k$-pullback of the strategy of $\overline{\mathcal{R}}$. Hence $\Sigma \in \Omega$ since $\Xi \in \Omega$. Σ acts on trees above $\delta_{\lambda^\mathcal{Q}}$ and moves the predicate for $\oplus_{\gamma<\lambda^\mathcal{Q}} \Psi_\gamma$ correctly by by our assumption on Ξ and branch condensation of $\oplus_{\gamma<\lambda^\mathcal{Q}} \Psi_\gamma$. These properties of Σ imply that $\mathcal{Q} \lhd \mathrm{Lp}^{\Omega, \oplus_{\gamma<\lambda^\mathcal{Q}} \Psi_\gamma}(\bigcup_{\gamma<\lambda^\mathcal{Q}} \mathcal{Q}_\gamma)$. Contradiction. For the case that there are $\alpha < \lambda^\mathcal{R}$, $\delta^\mathcal{R}_\alpha \leq \eta < \delta^\mathcal{R}_{\eta+1}$, and η is a strong cutpoint of $\mathcal{R}$, and $\mathcal{M}$ is a sound $\Psi_{\mathcal{R}(\alpha)}$-mouse projecting to η with iteration strategy in Ω, the proof is the same as that of Theorem 3.7.6 in [3].

Now we show Λ has branch condensation (see Figure 1). The proof of this comes from private conversations between the author and John Steel. We'd like to thank him for this. For notational simplicity, we write Λ^- for $\oplus_{\gamma<\lambda^\mathcal{Q}} \Psi_\gamma$. Hence, $\Lambda \notin \Omega$ and $\Lambda^- \in \Omega$. Suppose Λ does not have branch condensation. We have a minimal counterexample as follows: there are an iteration $i : \mathcal{Q} \to \mathcal{R}$ by Λ, a normal tree $\mathcal{U}$ on $\mathcal{R}$ in the window $[\xi, \gamma)$ where $\xi < \gamma$ are two consecutive Woodins in $\mathcal{R}$ such that $\sup i''\delta_{\lambda^\mathcal{Q}} \leq \xi$, two distinct cofinal branches of $\mathcal{U}$: b and $c = \Lambda_\mathcal{R}(\mathcal{U})$, an iteration map $j : \mathcal{Q} \to \mathcal{S}$ by Λ, and a map $\sigma : \mathcal{M}^\mathcal{U}_b \to \mathcal{S}$ such that $j = \sigma \circ i^\mathcal{U}_b \circ i$. We may also assume that if $\overline{\mathcal{R}}$ is the first model along the main branch of the stack from $\mathcal{Q}$ to $\mathcal{R}$ giving rise to i and $i_{\overline{\mathcal{R}},\mathcal{R}} : \overline{\mathcal{R}} \to \mathcal{R}$ be the natural map such that $i_{\overline{\mathcal{R}},\mathcal{R}}(\overline{\xi}) = \xi$ and $i_{\overline{\mathcal{R}},\mathcal{R}}(\overline{\gamma}) = \gamma$, then the extenders used to get from $\mathcal{Q}$ to $\overline{\mathcal{R}}$ have generators below $\overline{\xi}$. This gives us $\sup(Hull^\mathcal{R}(\xi \cup \{p\}) \cap \gamma) = \gamma$ where p is the standard parameter of $\mathcal{R}$. Let $\Phi = \Lambda^\sigma_\mathcal{S}$ and $\Phi^- = \oplus_{\xi<\lambda^{\mathcal{M}^\mathcal{U}_b}} \Phi_{\mathcal{M}^\mathcal{U}_b(\xi)}$. It's easy to see that $\Phi^- \in \Omega$. By the same proof as in the previous paragraph, Φ is Ω-fullness preserving. This of course implies that $\mathcal{M}^\mathcal{U}_b$ is Ω-full and $\Phi \notin \Omega$.

Now we compare $\mathcal{M}^\mathcal{U}_b$ and $\mathcal{M}^\mathcal{U}_c$. First we line up the strategies of $\mathcal{M}^\mathcal{U}_b|\delta(\mathcal{U})$ and $\mathcal{M}^\mathcal{U}_c|\delta(\mathcal{U})$ by iterating them into the (Ω-full) hod pair construction of some N^*_y (where y codes $(x, \mathcal{M}^\mathcal{U}_c, \mathcal{M}^\mathcal{U}_b)$). This can be done because the strategies of $\mathcal{M}^\mathcal{U}_b|\delta(\mathcal{U})$ and of $\mathcal{M}^\mathcal{U}_c|\delta(\mathcal{U})$ have branch condensation by Theorems 2.7.6 and 2.7.7 of [3][14]. This process produces a single normal tree

[14]We note here that suppose $(\mathcal{P}, \Sigma)$ is a hod pair and $\mathcal{P} \vDash \delta^\mathcal{P}$ has measurable cofinality. Then knowing that all "lower level" strategies of all iterates of $(\mathcal{P}, \Sigma)$ has branch condensation does not tell us that Σ itself

$$\begin{tikzcd}
\mathcal{Q} \arrow[r, "i"] \arrow[ddrr, "j"'] & \mathcal{R} \arrow[r, "\mathcal{U},c"] \arrow[dr, "\mathcal{U},b"'] & \mathcal{M}_c^{\mathcal{U}} \arrow[r, "\mathcal{W},d"] & \mathcal{M}_d^{\mathcal{W}} \arrow[r, "\tau"] & \mathcal{K} \\
 & & \mathcal{M}_b^{\mathcal{U}} \arrow[r, "\mathcal{W},a"] \arrow[d, "\sigma"] & \mathcal{M}_a^{\mathcal{W}} \arrow[ur, "\pi"] & \\
 & & \mathcal{S} & &
\end{tikzcd}$$

Figure 1: The proof of branch condensation of Λ in Lemma 3.2

$\mathcal{W}$. Let $a = \Phi(\mathcal{W})$ and $d = \Lambda_{\mathcal{M}_c^{\mathcal{U}}}(\mathcal{W})$. Let $X = Hull^{\mathcal{R}}(\xi \cup \{p\}) \cap \gamma$. Note that $(i_a^{\mathcal{W}} \circ i_b^{\mathcal{U}})$"X $\subseteq \delta(\mathcal{W})$ and $i_d^{\mathcal{W}} \circ i_c^{\mathcal{U}}$"X $\subseteq \delta(\mathcal{W})$. Now continue lining up $\mathcal{M}_a^{\mathcal{W}}$ and $\mathcal{M}_d^{\mathcal{W}}$ above $\delta(\mathcal{W})$ (using the same process as above). We get $\pi : \mathcal{M}_a^{\mathcal{W}} \to \mathcal{K}$ and $\tau : \mathcal{M}_d^{\mathcal{W}} \to \mathcal{K}$ (we indeed end up with the same model $\mathcal{K}$ by our assumption on the pair (Λ, Λ^-)). But then

$$(\pi \circ i_a^{\mathcal{W}} \circ i_b^{\mathcal{U}})\text{"X} = (\tau \circ i_d^{\mathcal{W}} \circ i_c^{\mathcal{U}})\text{"X}.$$

But by the fact that $(i_a^{\mathcal{W}} \circ i_b^{\mathcal{U}})$"X $\subseteq \delta(\mathcal{W})$ and $i_d^{\mathcal{W}} \circ i_c^{\mathcal{U}}$"X $\subseteq \delta(\mathcal{W})$ and π agrees with τ above $\delta(\mathcal{W})$, we get

$$(i_a^{\mathcal{W}} \circ i_b^{\mathcal{U}})\text{"X} = (i_d^{\mathcal{W}} \circ i_c^{\mathcal{U}})\text{"X}.$$

This gives $ran(i_a^{\mathcal{W}}) \cap ran(i_d^{\mathcal{W}})$ is cofinal in $\delta(\mathcal{W})$, which implies $a = d$. This in turns easily implies $b = c$. Contradiction. Finally, let $\mathcal{R}$ and $\mathcal{S}$ be Λ-iterates of $\mathcal{Q}$ and let $\Lambda_{\mathcal{R}}$ and $\Lambda_{\mathcal{S}}$ be the tails of Λ on $\mathcal{R}$ and $\mathcal{S}$ respectively. We want to show that $\mathcal{R}$ and $\mathcal{S}$ can be further iterated (using $\Lambda_{\mathcal{R}}$ and $\Lambda_{\mathcal{S}}$ respectively) to the same model. To see this, we compare $\mathcal{R}$ and $\mathcal{S}$ against the Ω-full hod pair construction of some N_y^* (for some y coding $(x, \mathcal{R}, \mathcal{S})$). Then during the comparison, only $\mathcal{R}$ and $\mathcal{S}$ move (to say $\mathcal{R}^*$ and $\mathcal{S}^*$). It's easy to see that $\mathcal{R}^* = \mathcal{S}^*$ and their strategies are the same (as the induced strategy of N_y^* on its appropriate background construction). □

By the properties of $(\mathcal{Q}, \Psi)$ and Λ, we get that $\rho(\mathcal{M}_\infty(\mathcal{Q}, \Lambda)) \leq \Theta$ and $(\text{HOD}|\Theta)^N = \mathcal{M}_\infty(\mathcal{Q}, \Lambda)|\Theta$. Let k be the least such that $\rho_{k+1}(\mathcal{Q}) \leq \delta_{\lambda^{\mathcal{Q}}}$.

Claim1. $\mathcal{M}_\infty(\mathcal{Q}, \Lambda) \notin N$.

Proof. Suppose not. Let $i : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the direct limit map according to Λ. By an absoluteness argument (i.e. using the absoluteness of the illfoundedness of the tree built in

has branch condensation.

$N[g]$ for $g \subseteq Col(\omega, |\mathcal{M}_\infty(\mathcal{Q}, \Lambda)|)$ generic over N of approximations of a embedding from $\mathcal{Q}$ into $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ extending the iteration embedding according to $\oplus_{\beta<\lambda^{\mathcal{Q}}}\Psi_\beta$ on $\mathcal{Q}|\delta_{\lambda^{\mathcal{Q}}})$, we get a map π such that

1. $\pi \in N$
2. $\pi : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$;
3. for each $\beta < \lambda^{\mathcal{Q}}$, $\pi|Q(\beta)$ is according to Ψ_β.
4. $\pi(p) = i(p)$ where $p = p_k(\mathcal{Q})$.

This implies that $\pi = i \in N$ since $\mathcal{Q}$ is $\delta_{\lambda^{\mathcal{Q}}}$-sound and $\rho(\mathcal{Q}) \leq \delta^{\lambda^{\mathcal{Q}}}$. But this map determines Λ in N as follows: let $\mathcal{T} \in N$ be countable and be according to Λ, N can build a tree searching for a cofinal branch b of $\mathcal{T}$ along with an embedding $\sigma : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ such that $\pi = \sigma \circ i_b^{\mathcal{T}}$. Using the fact that Λ has branch condensation, we easily get that $\Lambda \in N$. But this is a contradiction. □

Returning to the proof of (2), let $j =_{def} j_\mu : \mathrm{HOD} \to Ult_0(\mathrm{HOD}, \mu)$ and $\mathcal{W} = j(\langle \mathcal{M}_\beta \mid \beta < \alpha, \beta \text{ is limit}\rangle)(\alpha)$. Let $i : \mathcal{M}_\infty(\mathcal{Q}, \Lambda) \to Ult_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda), \mu)$ be the canonical map. Note that $A \notin \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. To see this, assume not, let $\mathcal{R} \lhd \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the first level $\mathcal{S}$ of $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ such that A is definable over $\mathcal{S}$.

Claim 2. $\mathcal{R} \in N$.

Proof. Recall that $\mathcal{W}$ is the first level of $\mathcal{M}_\infty^+$ such that $j(A) \cap \Theta$ is definable over $\mathcal{W}$. Now let

$$k : \mathcal{R} \to Ult_0(\mathcal{R}, \mu) =_{def} \mathcal{R}^*$$

be the Σ_0-ultrapower map. By the definition of $\mathcal{W}$ and $\mathcal{R}^*$ and the fact that they are both countably iterable, we get that $\mathcal{W} = \mathcal{R}^* \in N$. Let p be the standard parameters for $\mathcal{R}$. In N, we can compute $Th_0^{\mathcal{R}}(\Theta \cup p)$ as follows: for a formula ψ in the language of hod premice and $s \in \Theta^{<\omega}$,

$$(\psi, s) \in Th_0^{\mathcal{R}}(\Theta \cup p) \Leftrightarrow (\psi, j(s)) \in Th_0^{\mathcal{R}^*}(\Theta \cup k(s)).$$

Since $Th_0^{\mathcal{R}^*}(\Theta \cup k(s)) = Th_0^{\mathcal{W}}(\Theta \cup k(s)) \in N$, $j|\Theta \in N$, and $k(s) \in \mathcal{W} \in N$, we get $Th_0^{\mathcal{R}}(\Theta \cup p) \in N$. This shows $\mathcal{R} \in N$. □

To get a contradiction, we show $\mathcal{R} \lhd \mathcal{N}_\infty^+$ by showing $\mathcal{R}$ is satisfies property $(*)$ in N. Let $\mathcal{K}$ be a countable mouse embeddable into $\mathcal{R}$ by a map $k \in N$. Then we can compare $\mathcal{K}$ and $\mathcal{Q}$ against the Ω-full hod pair construction of some N_y^* just like in the argument on page 15; hence we may assume $\mathcal{K} \lhd \mathcal{Q}$ ($\mathcal{Q} \unlhd \mathcal{K}$ can't happen because then $\Lambda \in N$). The minimality assumption on $\mathcal{Q}$ easily implies $\mathcal{K} \lhd \mathrm{Lp}^{\Omega, \oplus_{\gamma<\lambda^{\mathcal{Q}}} \Psi_\gamma}(\mathcal{Q}|\delta_{\lambda^{\mathcal{Q}}})$. But then N can iterate $\mathcal{K}$ for stacks on $\mathcal{K}$ above $\delta_{\lambda^{\mathcal{Q}}} = \delta_{\lambda^{\mathcal{K}}}$, which is what we want to show. The fact that $\mathcal{R} \lhd \mathcal{N}^+_\infty$ contradicts $A \notin \mathcal{N}^+_\infty$.

Next, we note that $Ult_0(\mathrm{HOD}, \mu)|\Theta = Ult_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda), \mu)|\Theta$ and $i|\Theta = j|\Theta$. Let $\mathcal{R} = Th^{\mathcal{M}_\infty(\mathcal{Q},\Lambda)}(\Theta \cup \{p\})$ where $p = p_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda))$ and $\mathcal{S} = Th^{Ult_k(\mathcal{M}_\infty(\mathcal{Q},\Lambda),\mu)}(\Theta \cup \{i(p)\})$. We have that $\mathcal{M}_\alpha$ and $\mathcal{S}$ are sound hybrid mice in the same hierarchy, hence by countable iterability, we can conclude either $\mathcal{M}_\alpha \lhd \mathcal{S}$ or $\mathcal{S} \unlhd \mathcal{M}_\alpha$.

If $\mathcal{M}_\alpha \lhd \mathcal{S}$, then $\mathcal{M}_\alpha \in Ult_k(\mathcal{M}_\infty(\mathcal{Q}, \Lambda), \mu)$. This implies $A \in \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ by a computation similar to that in the proof of (1), i.e.

$$\beta \in A \Leftrightarrow \mathcal{M}_\infty(\mathcal{Q}, \Lambda) \vDash (i|\Theta)(\beta) \in \mathcal{M}_\alpha.$$

This is a contradiction to the fact that $A \notin \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. Now suppose $\mathcal{S} \unlhd \mathcal{M}_\alpha$. This then implies $\mathcal{S} \in Ult_0(\mathrm{HOD}, \mu)$, which in turns implies $\mathcal{M}_\infty(\mathcal{Q}, \Lambda) \in \mathrm{HOD}$ by the following computation: for any formula ϕ and $s \in \Theta^{<\omega}$,

$$(\phi, s) \in \mathcal{R} \Leftrightarrow \mathrm{HOD} \vDash (\phi, (j|\Theta)(s)) \in \mathcal{S}.$$

This is a contradiction to the claim. This completes the proof of (2). □

Theorem 1.2 completes our analysis of HOD for AD^+ models of the form "$V = L(\wp(\mathbb{R})) + \Theta = \theta_\alpha$ for some limit α" below "$\mathsf{AD}_\mathbb{R} + \Theta$ is regular."

4 The general successor case

Again, we assume $(*)$. Assume also that $\Theta = \theta_0$ or $\Theta = \theta_{\alpha+1}$ for some α. We prove Theorem 1.3 through a series of Lemmas in this section. The proof makes use of ideas from [6] and [9].

First we need to compute $\mathrm{HOD}|\Theta$. By [3, Section 4.3], if $\Theta > \theta_0$, then there is a hod pair $(\mathcal{P}, \Sigma)$ such that

(1) Σ is fullness preserving and has branch condensation;

(2) $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)|\theta_\alpha = \mathrm{HOD}|\theta_\alpha$, where $\mathcal{M}^+_\infty(\mathcal{P}, \Sigma)$ is the direct limit of all Σ-iterates of $\mathcal{P}$.

By [5, Theorem 0.1], we may chose $(\mathcal{P}, \Sigma)$ as above such that

(3) $V = L(\mathrm{Lp}^\Sigma(\mathbb{R}))$ (in the case $\Theta = \theta_0$, $(\mathcal{P}, \Sigma) = (\emptyset, \emptyset)$ and $V = L(\mathrm{Lp}(\mathbb{R}))$.

It is clear that there is no hod pair $(\mathcal{P},\Sigma)$ satisfying (1) and (2) above when $\mathrm{HOD}|\theta_\alpha$ is replaced by $\mathrm{HOD}|\Theta$ as this would imply that $\Sigma\notin V$. So to compute $\mathrm{HOD}|\Theta$, we need to mimic the computation in [9, Section 7] or [6, Section 7]. The main idea is to use Σ_1-reflection to produce a "next hod pair" $(\mathcal{P},\Sigma)$ satisfying (1)-(3) above with respect to $\mathrm{HOD}|\Theta$ of a reflected universe (this idea originates from Woodin).

Definition 4.1 (n-suitable pair)**.** *$(\mathcal{P},\Sigma)$ is an n-suitable pair if there is δ such that $(\mathcal{P}|(\delta^{+\omega})^{\mathcal{P}},\Sigma)$ is a hod pair and*

1. *$\mathcal{P}\vDash \mathsf{ZFC}$ - Replacement + "there are n Woodin cardinals, $\eta_0<\eta_1<...<\eta_{n-1}$ above δ";*
2. *$o(\mathcal{P})=sup_{i<\omega}(\eta_{n-1})^{+i^P}$;*
3. *$\mathcal{P}$ is a Σ-mouse over $\mathcal{P}|\delta$;*
4. *for any $\mathcal{P}$-cardinal $\eta>\delta$, if η is a strong cutpoint then $\mathcal{P}|(\eta^+)^{\mathcal{P}}=Lp^{\Sigma}(\mathcal{P}|\eta)$.*

For $\mathcal{P},\delta$ as in the above definition, let $\mathcal{P}^-=\mathcal{P}|(\delta^{+\omega})^{\mathcal{P}}$ and

$$\begin{aligned}\mathbb{B}(\mathcal{P}^-,\Sigma)=\{B\subseteq\wp(\mathbb{R})\times\mathbb{R}\times\mathbb{R}\quad &|\ B \text{ is } OD, \text{ for any } (\mathcal{Q},\Lambda) \text{ iterate of } (\mathcal{P}^-,\Sigma),\\ &\text{and for any } (x,y)\in B_{(\mathcal{Q},\Lambda)}, x \text{ codes } \mathcal{Q}\}.\end{aligned}$$

In the above definition, we identify Λ with the set of reals $\mathrm{Code}(\Lambda)$. We also write "$\mathcal{P}$ is Σ-n-suitable" for "$(\mathcal{P},\Sigma)$ is an n-suitable pair" . If $(\mathcal{P}^-,\Sigma)=(\emptyset,\emptyset)$, then each $B\in\mathbb{B}(\emptyset,\emptyset)$ can be canonically identified with an OD set of reals and hence $\mathbb{B}(\emptyset,\emptyset)$ can be canonically identified with the collection of OD sets of reals. Suppose $B\in\mathbb{B}(\mathcal{P}^-,\Sigma)$ and $\kappa<o(\mathcal{P})$. Let $\tau^{\mathcal{P}}_{B,\kappa}$ be the canonical term in $\mathcal{P}$ that captures B at κ i.e. for any $g\subseteq Col(\omega,\kappa)$ generic over $\mathcal{P}$

$$B_{(\mathcal{P}^-,\Sigma)}\cap\mathcal{P}[g]=(\tau^{\mathcal{P}}_{B,\kappa})_g.$$

For each $m<\omega$, let

$$\gamma^{\mathcal{P},\Sigma}_{B,m}=\sup(H_1^{\mathcal{P}}(\tau^{\mathcal{P}}_{B,(\eta_{n-1}^{+m})^{\mathcal{P}}})\cap\eta_0),$$

$$H^{\mathcal{P},\Sigma}_{B,m}=H_1^{\mathcal{P}}(\gamma^{\mathcal{P},\Sigma}_{B,m}\cup\{\tau^{\mathcal{P}}_{B,(\eta_{n-1}^{+m})^{\mathcal{P}}}\}),$$

$$\gamma^{\mathcal{P},\Sigma}_{B}=\sup\nolimits_{m<\omega}\gamma^{\mathcal{P},\Sigma}_{B,m},$$

and

$$H^{\mathcal{P},\Sigma}_{B}=\bigcup_{m<\omega}H^{\mathcal{P},\Sigma}_{B,m}.$$

Similar definitions can be given for $\gamma^{\mathcal{P},\Sigma}_{\vec{B},m}, H^{\mathcal{P},\Sigma}_{\vec{B},m}, \gamma^{\mathcal{P},\Sigma}_{\vec{B}}, H^{\mathcal{P},\Sigma}_{\vec{B}}$ for any finite sequence $\vec{B} \in \mathbb{B}(\mathcal{P}^-, \Sigma)$. One just needs to include relevant terms for each element of $\vec{B}$ in each relevant hull. The usual notions of B-iterability, strong B-iterability, and the corresponding weak iteration games $\mathcal{WG}(\mathcal{P}, \Sigma)$, $\mathcal{WG}(\mathcal{P}, \Sigma, B)$ are defined in [3, Section 3.1]. Now we're ready to define our direct limit system. Let

$$\mathcal{F} = \{(\mathcal{P}, \Sigma, \vec{B}) \quad | \quad \vec{B} \in \mathbb{B}(\mathcal{P}^-, \Sigma)^{<\omega}, (\mathcal{P}^-, \Sigma) \text{ satisfies } (1)-(3), (\mathcal{P}, \Sigma) \text{ is } n\text{-suitable}$$
$$\text{for some } n, \text{ and } (\mathcal{P}, \Sigma) \text{ is strongly } \vec{B}\text{-iterable}\}.$$

The ordering on $\mathcal{F}$ is defined as follows:

$$\begin{aligned}(\mathcal{P}, \Sigma, \vec{B}) \preccurlyeq (\mathcal{Q}, \Lambda, \vec{C}) \quad \Leftrightarrow \quad & \vec{B} \subseteq \vec{C}, \exists r (r \text{ is a run of } \mathcal{WG}(\mathcal{P}, \Sigma, \vec{B}) \text{ with the last model } \mathcal{P}^* \\ & \text{such that } (\mathcal{P}^*)^- = \mathcal{Q}^-,\ \Sigma_{(\mathcal{P}^*)^-} = \Lambda, \mathcal{P}^* = \mathcal{Q}|(\eta^{+\omega})^{\mathcal{Q}} \\ & \text{ where } \mathcal{Q} \vDash \eta > o(\mathcal{Q}^-) \text{ is Woodin}).\end{aligned}$$

Suppose $(P, \Sigma, \vec{B}) \preccurlyeq (Q, \Lambda, \vec{C})$ then there is a unique map $\pi^{(\mathcal{P},\Sigma),(\mathcal{Q},\Delta)}_{\vec{B}} : H^{\mathcal{P},\Sigma}_{\vec{B}} \to H^{\mathcal{Q},\Lambda}_{\vec{B}}$ given by strong $\vec{B}$-iterability. $(\mathcal{F}, \preccurlyeq)$ is then directed. Let

$$\mathcal{M}_\infty = \text{direct limit of } (\mathcal{F}, \preccurlyeq) \text{ under maps } \pi^{(\mathcal{P},\Sigma),(\mathcal{Q},\Delta)}_{\vec{B}}.$$

Also for each $(\mathcal{P}, \Sigma, \vec{B}) \in \mathcal{F}$, let

$$\pi^{(\mathcal{P},\Sigma),\infty}_{\vec{B}} : H^{\mathcal{P},\Sigma}_{\vec{B}} \to \mathcal{M}_\infty$$

be the natural map.

Clearly, $\mathcal{M}_\infty \subseteq \mathrm{HOD}$. But first, we need to show $\mathcal{F} \neq \emptyset$. In fact, we prove a stronger statement.

Theorem 4.2. *Suppose $(\mathcal{P}, \Sigma)$ satisfies (1)-(3)[15]. Let $B \in \mathbb{B}(\mathcal{P}, \Sigma)$. Then for each $1 \leq n < \omega$, there is a $\mathcal{Q}$ such that $\mathcal{Q}^-$ is a Σ-iterate of $\mathcal{P}^-$, $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-})$ is n-suitable and $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-}, B) \in \mathcal{F}$.*

Proof. Suppose not. By Σ_1-reflection ([8, Theorem 1]), there is an transitive model N coded by a Suslin, co-Suslin set of reals such that $Code(\Sigma) \in \wp(\mathbb{R})^N$ and

$$\begin{aligned} N \ \vDash \ & \mathsf{ZF}^- + \mathsf{AD}^+ + S\mathsf{MC} + \text{“}\Theta \text{ exists and is successor in the Solovay sequence ”} + \\ & \text{“}\exists B \in \mathbb{B}(\mathcal{P}, \Sigma)(\nexists \mathcal{Q}, n)((\mathcal{Q}, \Sigma) \text{ is } n\text{-suitable and } (\mathcal{Q}, \Sigma, B) \in \mathcal{F})\text{”}.\end{aligned}$$

[15] From now on, we use this to mean: either $\Theta > \theta_0$ and $(\mathcal{P}, \Sigma)$ satisfies (1)-(3) or $(\mathcal{P}, \Sigma) = (\emptyset, \emptyset)$.

We take a minimal such N and fix a $B \in \mathbb{B}(\mathcal{P}, \Sigma)^N$ witnessing the failure of the Theorem in N. Using [13, Theorem 10.3] and the assumption on N, there is an $x \in \mathbb{R}$ and a tuple $\langle N_x^*, \delta_x, \Sigma_x \rangle$ satisfying the conclusions of Theorem [13, Theorem 10.3] relative to Γ- a good pointclass containing $(\wp(\mathbb{R})^N, N's$ first order theory). Futhermore, let's assume that N_x^* Suslin captures $\langle A \mid A$ is projective in $\Sigma \rangle$). Let $\Omega = \wp(\mathbb{R})^N$. For simplicity, we show that in N, there is a Σ-iterate $(\mathcal{R}, \Sigma_{\mathcal{R}})$ such that there is a 1-suitable $(\mathcal{S}, \Sigma_{\mathcal{R}})$ such that $(\mathcal{S}, \Sigma_{\mathcal{R}}, B) \in \mathcal{F}$.

By the assumption on N, $N \vDash V = \mathrm{Lp}^{\Sigma}(\mathbb{R})$. Now N_x^* has club many $(\Sigma_1^2(\Sigma))^{\Omega}$ Woodins below δ_x by a standard argument (see [10]). By performing the Ω-hod pair construction in N_x^* and iterating $(\mathcal{P}, \Sigma)$ into this construction, we may assume $(\mathcal{P}, \Sigma)$ comes from the Ω-hod pair construction of N_x^*. Now, the full background construction $L[E, \Sigma][\mathcal{P}]$ done in N_x^* will reach a model having ω Woodins (which are the first ω $(\Sigma_1^2(\Sigma))^{\Omega}$ Woodins in N_x^*) and projecting across the sup of its first ω Woodins. Let $\mathcal{Q}$ be the first model in the construction with that property. By coring down if necessary, we may assume that $\mathcal{Q}$ is sound. Let $\langle \delta_i^{\mathcal{Q}} \mid i < \omega \rangle$ be the first ω Woodins of $\mathcal{Q}$ above $o(\mathcal{P})$. A similar self-explanatory notation will be used to denote the Woodins of any Λ-iterate of $\mathcal{Q}$. Hence $\rho_\omega(\mathcal{Q}) < \sup_{i<\omega} \delta_i$. Let Λ (which extends Σ) be the strategy of $\mathcal{Q}$ induced from the background universe. Λ is Ω-fullness preserving and has the Dodd-Jensen property. The following lemma shows that an iterate of Λ is strongly B-iterable and in fact it shows a bit more.

Lemma 4.3. *There is an iterate $(\mathcal{R}, \Lambda_{\mathcal{R}})$ of $(\mathcal{Q}, \Lambda)$ with strong B-condensation in that if $i : \mathcal{R} \to \mathcal{S}$ is according to $\Lambda_{\mathcal{R}}$ and below $\delta_0^{\mathcal{Q}}$ and $j : \mathcal{R} \to \mathcal{W}$ is such that there is a $k : \mathcal{W} \to \mathcal{S}$ such that $i = k \circ j$ then $i(\tau^{\mathcal{R}}_{B,\delta_0^{\mathcal{R}}}) = \tau^{\mathcal{S}}_{B,\delta_0^{\mathcal{S}}}$, $\mathcal{W}$ is Ω-full, and $k^{-1}(\tau^{\mathcal{S}}_{B,\delta_0^{\mathcal{S}}}) = \tau^{\mathcal{W}}_{B,j(\delta_0^{\mathcal{R}})}$.*

Proof. That we get $\mathcal{W}$ being Ω-full is easy because $\Lambda \notin N$. So we only need to prove the other clauses. Suppose not. Using the property of $\mathcal{Q}$ and the relativized (to Σ) Prikry forcing $\mathbb{P}$ in N, we get that for any n, there is an iterate $\mathcal{R}$ of $\mathcal{Q}$ (above $\delta_0^{\mathcal{Q}}$) extending a Prikry generic and having N as the (new) derived model (computed at the sup of the first ω Woodins above $o(\mathcal{P})$). Furthermore, this property holds for any Λ iterate of $\mathcal{Q}$. Without going further into details of the techniques used in [5], we remark that if $\mathcal{R}$ is an $\mathbb{R}$-genericity iterate of $\mathcal{Q}$, then the new derived model of $\mathcal{R}$ is N. In other words, once we know one such $\mathbb{R}$-genericity iterate of $\mathcal{Q}$ realizes N as its derived model then all $\mathbb{R}$-genericity iterates of $\mathcal{Q}$ do. Let (ϕ, s) define B over N, i.e.

$$(\mathcal{R}, \Psi, x, y) \in B \;\Leftrightarrow\; N \vDash \phi[((\mathcal{R}, \Psi, x, y)), s].$$

The following argument mirrors that of Lemma 3.2.15 in [3] though it's not clear to the author who this argument is orginially due to. The process below is described in Figure 2.

From now to the end of the proof, all stacks on $\mathcal{Q}$ or its iterates thereof are below the $\delta_0^{\mathcal{Q}}$ or its image. By our assumption, there is $\langle \vec{\mathcal{T}}_i, \vec{\mathcal{S}}_i, \mathcal{Q}_i, \mathcal{R}_i, \pi_i, \sigma_i, j_i \mid i < \omega \rangle \in N$ such that

1. $\mathcal{Q}_0 = \mathcal{Q}$; $\vec{\mathcal{T}}_0$ is a stack on $\mathcal{Q}$ according to Λ with last model $\mathcal{Q}_1$; $\pi_0 = i^{\vec{\mathcal{T}}_0}$; $\vec{\mathcal{S}}_0$ is a stack on $\mathcal{Q}$ with last model $\mathcal{R}_0$; $\sigma_0 = i^{\vec{\mathcal{S}}_0}$; and $j_0 : \mathcal{R}_0 \to \mathcal{Q}_1$.

2. $\vec{\mathcal{T}}_i$ is a stack on $\mathcal{Q}_i$ according to Λ with last model $\mathcal{Q}_{i+1}$; $\pi_i = i^{\vec{\mathcal{T}}_i}$; $\vec{\mathcal{S}}_i$ is a stack on $\mathcal{Q}_i$ with last model $\mathcal{R}_i$; $\sigma_i = i^{\vec{\mathcal{S}}_i}$; $j_0 : \mathcal{R}_i \to \mathcal{Q}_{i+1}$.

3. for all k, $\pi_k = j_k \circ \sigma_k$.

4. for all k, $\pi_k(\tau^{\mathcal{Q}_k}_{B,\delta_0^{\mathcal{Q}_k}}) \neq \tau^{\mathcal{Q}_{k+1}}_{B,\delta_0^{\mathcal{Q}_{k+1}}}$ or $j_k(\tau^{\mathcal{R}_k}_{B,\delta_0^{\mathcal{R}_k}}) \neq \tau^{\mathcal{Q}_{k+1}}_{B,\delta_0^{\mathcal{Q}_{k+1}}}$.

Let $\mathcal{Q}_\omega$ be the direct limit of the $\mathcal{Q}_i$'s under maps π_i's. First we rename $\langle \mathcal{Q}_i, \mathcal{R}_i, \pi_i, \sigma_i, j_i \mid i < \omega \rangle$ into $\langle \mathcal{Q}_i^0, \mathcal{R}_i^0, \pi_i^0, \sigma_i^0, j_i^0 \mid i < \omega \rangle$. We fix in $V^{Col(\omega,\mathbb{R})}$ $\langle x_i \mid i < \omega \rangle$, a generic enumeration of $\mathbb{R}$[16]. Using our assumption on $\mathcal{Q}$, we get $\langle \mathcal{Q}_i^n, \mathcal{R}_i^n, \pi_i^n, \sigma_i^n, j_i^n, \tau_1^n, k_i^n \mid n, i \leq \omega \rangle$ such that

1. $\mathcal{Q}_i^\omega$ is the direct limit of the $\mathcal{Q}_i^n$'s under maps τ_i^n's for all $i \leq \omega$.

2. $\mathcal{R}_i^\omega$ is the direct limit of the $\mathcal{R}_i^n$'s under maps k_i^n's for all $i < \omega$.

3. $\mathcal{Q}_\omega^n$ is the direct limit of the $\mathcal{Q}_i^n$'s under maps π_i^n's.

4. for all $n \leq \omega$, $i < \omega$, $\pi_i^n : \mathcal{Q}_i^n \to \mathcal{Q}_{i+1}^n$; $\sigma_i^n : \mathcal{Q}_i^n \to \mathcal{R}_i^n$; $j_i^n : \mathcal{R}_i^n \to \mathcal{Q}_{i+1}^n$ and $\pi_i^n = j_i^n \circ \sigma_i^n$.

5. Derived model of the $\mathcal{Q}_i^n$'s, $\mathcal{R}_i^n$'s is N.

Then we start by iterating $\mathcal{Q}_0^0$ above $\delta_0^{\mathcal{Q}_0^0}$ to $\mathcal{Q}_0^1$ to make x_0-generic at $\delta_1^{\mathcal{Q}_0^1}$. During this process, we lift the genericity iteration tree to all $\mathcal{R}_n^0$ for $n < \omega$ and $\mathcal{Q}_n^0$ for $n \leq \omega$. We pick branches for the tree on $\mathcal{Q}_0^0$ by picking branches for the lift-up tree on $\mathcal{Q}_\omega^0$ using $\Lambda_{\mathcal{Q}_\omega^0}$. Let $\tau_0^0 : \mathcal{Q}_0^0 \to \mathcal{Q}_0^1$ be the iteration map and $\mathcal{W}$ be the end model of the lift-up tree on $\mathcal{Q}_\omega^0$. We then iterate the end model of the lifted tree on $\mathcal{R}_0^0$ to $\mathcal{R}_0^1$ to make x_0 generic at $\delta_1^{\mathcal{R}_0^1}$ with branches being picked by lifting the iteration tree onto $\mathcal{W}$ and using the branches according to $\Lambda_{\mathcal{W}}$. Let $k_0^0 : \mathcal{R}_0^0 \to \mathcal{R}_0^1$ be the iteration embedding, $\sigma_0^1 : \mathcal{Q}_0^1 \to R_0^1$ be the natural map, and $\mathcal{X}$ be the end model of the lifted tree on the $\mathcal{W}$ side. We then iterate the end model of the lifted stack on $\mathcal{Q}_1^0$ to $\mathcal{Q}_1^1$ to make x_0 generic at $\delta_1^{\mathcal{Q}_1^1}$ with branches being picked by lifting the tree to $\mathcal{X}$ and using branches picked by $\Lambda_{\mathcal{X}}$. Let $\tau_1^0 : \mathcal{Q}_1^0 \to \mathcal{Q}_1^1$ be the iteration embedding, $j_0^1 : \mathcal{R}_0^1 \to \mathcal{Q}_1^1$ be the natural map, and $\pi_0^1 = j_0^1 \circ \sigma_0^1$. Continue this process of making x_0 generic for the later models $\mathcal{R}_n^0$'s and $\mathcal{Q}_n^0$'s for $n < \omega$. We then let $\mathcal{Q}_\omega^1$ be the

[16]We don't know that $\mathcal{Q}$ is iterable enough to do an $\mathbb{R}$-genericity iteration, but we can work with a countable substructure of N and enumerate the reals of this structure in V. $\mathcal{Q}$ then is sufficiently iterable for the upcoming argument.

$$
\begin{tikzcd}
\mathcal{Q}_0^0 \arrow[rr, "\pi_0^0"] \arrow[dr, "\sigma_0^0"] \arrow[dd, "\tau_0^0"] & & \mathcal{Q}_1^0 \arrow[rr, "\pi_1^0"] \arrow[dr, "\sigma_1^0"] \arrow[dd, "\tau_1^0"] & & \mathcal{Q}_2^0 \arrow[r, "\pi_2^0"] \arrow[dd, "\tau_2^0"] & \cdots & \mathcal{Q}_\omega^0 \arrow[dd, "\tau_\omega^0"] \\
& \mathcal{R}_0^0 \arrow[ur, "j_0^0"] \arrow[dd, "k_0^0"] & & \mathcal{R}_1^0 \arrow[ur, "j_1^0"] \arrow[dd, "k_1^0"] & & & \\
\mathcal{Q}_0^1 \arrow[rr, "\pi_0^1"] \arrow[dr, "\sigma_0^1"] \arrow[dd, "\tau_0^1"] & & \mathcal{Q}_1^1 \arrow[rr, "\pi_1^1"] \arrow[dr, "\sigma_1^1"] \arrow[dd, "\tau_1^1"] & & \mathcal{Q}_2^1 \arrow[r, "\pi_2^1"] \arrow[dd, "\tau_2^1"] & \cdots & \mathcal{Q}_\omega^1 \arrow[dd, "\tau_\omega^1"] \\
& \mathcal{R}_0^1 \arrow[ur, "j_0^1"] \arrow[dd, "k_0^1"] & & \mathcal{R}_1^1 \arrow[ur, "j_1^1"] \arrow[dd, "k_1^1"] & & & \\
\mathcal{Q}_0^2 \arrow[rr, "\pi_0^2"] \arrow[dr, "\sigma_0^2"] \arrow[dd, "\tau_0^2"] & & \mathcal{Q}_1^2 \arrow[rr, "\pi_1^2"] \arrow[dr, "\sigma_1^2"] \arrow[dd, "\tau_1^2"] & & \mathcal{Q}_2^2 \arrow[r, "\pi_2^2"] \arrow[dd, "\tau_2^2"] & \cdots & \mathcal{Q}_\omega^2 \arrow[dd, "\tau_\omega^2"] \\
& \mathcal{R}_0^2 \arrow[ur, "j_0^2"] \arrow[d, "k_0^2"] & & \mathcal{R}_1^2 \arrow[ur, "j_1^2"] \arrow[d, "k_1^2"] & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
\mathcal{Q}_0^\omega \arrow[rr, "\pi_0^\omega"] \arrow[dr, "\sigma_0^\omega"] & & \mathcal{Q}_1^\omega \arrow[rr, "\pi_1^\omega"] \arrow[dr, "\sigma_1^\omega"] & & \mathcal{Q}_2^\omega \arrow[r, "\pi_2^\omega"] & \cdots & \mathcal{Q}_\omega^\omega \\
& \mathcal{R}_0^\omega \arrow[ur, "j_0^\omega"] & & \mathcal{R}_1^\omega \arrow[ur, "j_1^\omega"] & & &
\end{tikzcd}
$$

Figure 2: The process in Theorem 4.2

direct limit of the $\mathcal{Q}_n^1$ under maps π_n^1's. We then start at $\mathcal{Q}_0^1$ and repeat the above process to make x_1 generic appropriate iterates of $\delta_2^{\mathcal{Q}_0^1}$ etc. This whole process define models and maps $\langle \mathcal{Q}_i^n, \mathcal{R}_i^n, \pi_i^n, \sigma_i^n, j_i^n, \tau_1^n, k_i^n \mid n, i \leq \omega \rangle$ as described above. See Figure 2.

Note that by our construction, for all $n < \omega$, the maps π_n^0's and τ_ω^n's are via Λ or its appropriate tails; furthermore, $\mathcal{Q}_\omega^\omega$ is wellfounded and full (with respect to mice in N). This in turns implies that the direct limits $\mathcal{Q}_n^\omega$'s and $\mathcal{R}_n^\omega$'s are wellfounded and full. We must then have that for some k, for all $n \geq k$, $\pi_n^\omega(s) = s$. This implies that for all $n \geq k$

$$\pi_n^\omega(\tau^{\mathcal{Q}_n^\omega}_{B,\delta_0^{\mathcal{Q}_n^\omega}}) = \tau^{\mathcal{Q}_{n+1}^\omega}_{B,\delta_0^{\mathcal{Q}_{n+1}^\omega}}.$$

We can also assume that for all $n \geq k$, $\sigma_n^\omega(s) = s, j_n^\omega(s) = s$. Hence

$$\sigma_n^\omega(\tau^{\mathcal{Q}_n^\omega}_{B,\delta_0^{\mathcal{Q}_n^\omega}}) = \tau^{\mathcal{R}_n^\omega}_{B,\delta_0^{\mathcal{R}_n^\omega}}.;$$

$$j_n^\omega(\tau^{\mathcal{R}_n^\omega}_{B,\delta_0^{\mathcal{R}_n^\omega}}) = \tau^{\mathcal{Q}_{n+1}^\omega}_{B,\delta_0^{\mathcal{Q}_{n+1}^\omega}}.;$$

This is a contradiction, hence we finish the proof of the lemma. □

The lemma easily implies a contradiction since we can just let our desired $\mathcal{S}$ be $\mathcal{R}|((\delta_0^{\mathcal{R}})^{+\omega})^{\mathcal{R}}$. Strong B-iterability of $\mathcal{S}$ inside N follows from the lemma and the fact that $\Lambda_{\mathcal{R}}$ has the Dodd-Jensen property. This finishes the proof of the theorem. □

Remark 4.4. The proof of Theorem 4.2 also shows that if $(\mathcal{P}, \Sigma)$ is n-suitable and $(\mathcal{P}, \Sigma, B) \in \mathcal{F}$ and $C \in \mathbb{B}(\mathcal{P}^-, \Sigma)$ then there is a B-iterate $\mathcal{Q}$ of $\mathcal{P}$ such that $(\mathcal{Q}, \Sigma_{\mathcal{Q}^-}, B \oplus C) \in \mathcal{F}$; in fact, $\mathcal{Q}$ has strong $B \oplus C$-condensation as defined in the proof of Theorem 4.2.

It is easy to see that $\mathcal{M}_\infty|\theta_\alpha = \mathrm{HOD}|\theta_\alpha$. Let $\langle \eta_i \mid i < \omega\rangle$ be the increasing enumeration of Woodin cardinals in $\mathcal{M}_\infty$ larger than θ_α. Theorem 4.2 is used to show that $\mathcal{M}_\infty$ is large enough in that

Lemma 4.5. *1. $\mathcal{M}_\infty$ is well-founded.*

2. $\mathcal{M}_\infty|\eta_0 = HOD|\Theta$. In particular, $\eta_0 = \Theta$.

Proof. We prove (1) and (2) simultaneously. For a similar argument, see Lemma 3.3.2 in [3]. Toward a contradiction, suppose not. By Σ_1-reflection (Theorem [8, Theorem 1]), there is a transitive model N coded by a Suslin, co-Suslin set of reals such that $Code(\Sigma) \in \wp(\mathbb{R})^N$ and

$$\begin{aligned} N \models\ & \mathsf{ZF}^- + \mathsf{DC} + \mathsf{SMC} + \text{“}\Theta \text{ exists and is successor in the Solovay sequence ”} + \\ & \text{“(1) and (2) do not both hold”}. \end{aligned}$$

We take a minimal such N and let $\Omega = \wp(\mathbb{R})^N$. We get $N \models V = \mathrm{Lp}^\Sigma(\mathbb{R})$ and a $(\mathcal{Q}, \Lambda)$ with the property that for all $B \in \mathbb{B}(P, \Sigma)^N$, there is a Λ iterate $\mathcal{R}$ of $\mathcal{Q}$ that with strong B-condensation (by Lemma 4.3). $(\mathcal{Q}, \Lambda)$ also has the property that any Λ iterate $\mathcal{R}$ of $\mathcal{Q}$ can be further iterated by $\Lambda_{\mathcal{R}}$ to $\mathcal{S}$ such that N is the derived model of $\mathcal{S}$.

Fix $\langle \alpha_i \mid i < \omega\rangle$ a cofinal in Θ^Ω sequence of ordinals. Such a sequence exists since $\Omega = Env((\Sigma_1^2(\Sigma))^N)$. For the rest of the proof, we write HOD for HOD^N, Θ for Θ^N etc. For each n, let

$$\begin{aligned} D_n = \{(\mathcal{R}, \Psi, x, y) \quad \mid \quad & (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R}, \\ & y \in \text{ the least } OD^N_\Psi \text{ set of reals with Wadge rank } \geq \alpha_n\} \end{aligned}$$

Clearly, for all n, $D_n \in \mathbb{B}(\mathcal{P}, \Sigma)^N$. Without loss of generality, we may assume Λ has strong D_n-condensation for each n. Let $\vec{D} = \langle D_n \mid n < \omega\rangle$. Before proving the next claim, let us introduce the following notion. First let for a set $A (A \subseteq \mathbb{R}$ or $A \in \mathbb{B}(\mathcal{P}, \Sigma))$, $\tau^{Q,0}_{A,m}$ be the canonical capturing term for A in $\mathcal{Q}$ at $(\delta_0^{+m})^{\mathcal{Q}}$. Set

$$\gamma^{\mathcal{Q},0}_{D_i,m} = sup\{H_1^{\mathcal{Q}}(P \cup \{\tau^{\mathcal{Q},0}_{D_i,m}\}) \cap \delta_0\};$$
$$\gamma^{\mathcal{Q},0}_{D_i} = \sup_{m<\omega} \gamma^{\mathcal{Q},0}_{D_i,m}.$$

Claim 1. For any Λ-iterate $(\mathcal{S}, \Upsilon)$ of $\mathcal{Q}$. Suppose $i : \mathcal{Q} \to \mathcal{S}$ is the itaration map. Then

$$i(\delta_0) = \sup_{i<\omega} \gamma^{\mathcal{S},0}_{D_i}.$$

Proof. Working in N, by the minimality of N, we can let $\langle A_i \mid i < \omega\rangle$ be a sequence of OD^N_Σ self-justifying system such that A_0 is a universal $\Sigma^2_1(\Sigma)$ set; $A_1 = \mathbb{R}\backslash A_0$ (see [14]). Suppose ϕ_i and $s_i \in \mathrm{OR}^{<\omega}$ are such that

$$x \in A_i \;\Leftrightarrow\; N \vDash \phi_i[\Sigma, s_i, x]$$

Now for each i, let

$$A_i^* = \{(\mathcal{R}, \Psi, x, y) \quad | \quad (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R}, \\ N \vDash \phi_i[\Psi, s_i, y]\}$$

Aside from the assumption about $(\mathcal{Q}, \Lambda)$ above, we also assume Λ is guided by $\langle A_i \mid i < \omega\rangle$ for stacks above $\mathcal{P}$ and below δ_0. This is possible by relativizing to Σ the proof of a similar fact in the case $\Theta = \theta_0$. This means

$$\delta_0 = \sup_{i<\omega} \gamma^{\mathcal{Q},0}_{A_i^*}.$$

This fact in turns implies

$$\delta_0 = \sup_{i<\omega} \gamma^{\mathcal{Q},0}_{D_i}.$$

To see this, fix an A_i^*. We show that there is a j such that $\gamma^{\mathcal{Q},0}_{D_j} \geq \gamma^{\mathcal{Q},0}_{A_i^*}$. Fix a real coding $\mathcal{P}$ and let j be such that

$$w(A_i) = w((A_i^*)_{(\mathcal{P},\Sigma,x)}) \leq w((D_j)_{(\mathcal{P},\Sigma,x)}).$$

Let z be a real witnessing the reduction. Then there is a map $i : \mathcal{Q} \to \mathcal{R}$ such that

1. i is according to Λ and the iteration is above $\mathcal{Q}^- = \mathcal{P}$;

2. z is generic for the extender algebra $\mathbb{A}$ of $\mathcal{R}$ at $\delta^{\mathcal{R}}$.

Note that $i(\tau^{\mathcal{Q}}_{A^*_i}) = \tau^{R}_{A^*_i}$, $i(\tau^{\mathcal{Q}}_{D_j}) = \tau^{\mathcal{R}}_{D_j}$, and $\mathcal{R}[z] \vDash \tau_{A^*_i} \leq_w \tau_{D_j}$ via z. Hence $\tau^{\mathcal{R}}_{A^*_i} \in X = \{\tau \in \mathcal{R}^{\mathbb{A}} \mid (\exists p \in \mathbb{A})(p \Vdash_R \tau \leq_w \tau_{D_j} \text{ via } \dot{z})\}$ and $|X|^{\mathcal{R}} < \delta^{\mathcal{R}}$ (by the fact that the extender algebra $\mathbb{A}$ is $\delta^{\mathcal{R}}$-cc). But X is definable over $\mathcal{R}$ from $\tau^{R}_{D_j}$, hence $|X|^{\mathcal{R}} < \gamma^{R,0}_{D_j}$. Since $\tau^{R}_{A^*_i} \in X$, $\gamma^{\mathcal{R},0}_{A^*_i} \leq \gamma^{\mathcal{R},0}_{D_j}$ which in turns implies $\gamma^{\mathcal{Q},0}_{A^*_i} \leq \gamma^{\mathcal{Q},0}_{D_j}$.

Now to finish the claim, let $(\mathcal{S}, \Upsilon)$ be a Λ iterate of $\mathcal{Q}$. Suppose $i : \mathcal{Q} \to \mathcal{S}$ is the iteration map. Let $\mathcal{R} = i(\mathcal{P})$ and $\Sigma_{\mathcal{Q}}$ be the tail of Σ under the iteration. We claim that

$$i(\delta_0) = \sup\nolimits_{i<\omega} \gamma^{S,0}_{D_i}. \qquad (\dagger)$$

This is easily seen to finish the proof of Claim 1. To see (†), we repeat the proof of the previous part applied to $(\mathcal{S}, \Upsilon)$ and $\langle B_i \mid i < \omega\rangle$, a $OD^N_{\Sigma_{\mathcal{Q}}}$ self-justifying system where B_0 is a universal $\Sigma^2_1(\Sigma_{\mathcal{Q}})$; $B_1 = \mathbb{R} \backslash B_0$. We may assume $(\mathcal{S}, \Upsilon)$ is guided by $\langle B_i \mid i < \omega\rangle$ for stacks above R and below $i(\delta_0)$. Now we are in the position to apply the exact same argument as above and conclude that (†) holds. Hence we're done. □

The proof of claim 1 shows Λ is guided by $\langle D_i \mid i < \omega\rangle$. Since Λ also has strong D_i-condensation for each i, Λ has branch condensation. Therefore, the direct limit $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ is defined and is wellfounded. This implies that in N, $\mathcal{M}_\infty$ is wellfounded. Let $\langle \delta_i \mid i < \omega\rangle$ be the first ω Woodins of $\mathcal{Q}$ above $\mathcal{Q}^-$ and $i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\infty} : \mathcal{Q} \to \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ be the iteration embedding according to Λ and $\langle \eta_n \mid n < \omega\rangle = \langle i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\infty}(\delta_i) \mid i < \omega\rangle$. For $(\mathcal{R}, \Lambda_{\mathcal{R}})$ and iterate of $(\mathcal{Q}, \Lambda)$, let $i^{\mathcal{R},\Lambda_{\mathcal{R}}}_{\mathcal{R},\infty}$ have the obvious meaning and $i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{R}}$ be the iteration map according to Λ. Note that in N, $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)|\eta_n = \mathcal{M}_\infty|\eta_n$ for all n.

Claim 2. $|\mathcal{M}_\infty(\mathcal{Q}, \Lambda)|\eta_0| = V^{\mathrm{HOD}}_{\eta_0}$. $\qquad$ (†)

Proof. To show (†), it is enough to show that if $A \subseteq \alpha < \eta_0$ and A is OD then $A \in \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. To see this, let i be such that $\gamma^{\mathcal{M}_\infty(\mathcal{Q},\Lambda),0}_{D_i} > \alpha$ (such an i exists by the proof of Claim 1). Let

$$\begin{aligned} C = \{(\mathcal{R}, \Psi, x, y) \quad \mid \quad & (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R},\ y \text{ codes } (N, \gamma) \\ & \text{such that } (\mathcal{N}, \Psi) \text{ is 1-suitable, } \mathcal{N} \text{ is strongly } D_i \text{ iterable via a} \\ & \text{quasi-strategy } \Phi \text{ extending } \Psi,\ \gamma < \gamma^{\mathcal{N},0}_{D_i},\ \pi^{(\mathcal{N},\Psi),\infty}_{D_i}(\gamma) \in A\}. \end{aligned}$$

By replacing $\mathcal{Q}$ by an iterate we may assume $(\mathcal{Q}, \Lambda)$ is C-iterable. Let $\tau^{\mathcal{Q}}_C = \tau^{\mathcal{Q}}_{C,(\delta_0^{+\omega})^{\mathcal{Q}}}$ and

$\tau_C = i^{(\mathcal{Q},\Lambda)}_{\mathcal{Q},\infty}(\tau^{\mathcal{Q}}_C)$. The following equivalence is easily shown by a standard computation:

$$\xi \in A \;\Leftrightarrow\; \mathcal{M}_\infty(\mathcal{Q},\Lambda) \vDash\!\Vdash_{Col(\omega,\eta_0^{+\omega})} \text{ "if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\infty}(\mathcal{P}), y \text{ codes } (\mathcal{M}_\infty(\mathcal{Q},\Lambda)|\eta_0^{+\omega},\xi) \text{ then } (x,y) \in \tau_C\text{"}.$$

For the reader's convenience, we show why the above equivalence holds. First suppose $\xi \in A$. Let $(\mathcal{S},\Xi) \in I(\mathcal{Q},\Lambda)$ be such that there is a $\gamma < \gamma^{\mathcal{S},0}_{D_i}$ and $i^{\mathcal{S},\Xi}_{\mathcal{S},\infty}(\gamma) = \xi$. Then we have (letting $\nu = i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\delta_0)$)

$$\mathcal{S} \vDash\!\Vdash_{Col(\omega,\nu^{+\omega})} \text{ "if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\mathcal{P}), y \text{ codes } (S|\nu^{+\omega},\gamma) \text{ then } (x,y) \in i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\tau^{\mathcal{Q},\Lambda}_C)\text{"}.$$

By applying $i^{\mathcal{S},\Xi}_{\mathcal{S},\infty}$ to this ,we get

$$\mathcal{M}_\infty(\mathcal{Q},\Lambda) \vDash\!\Vdash_{Col(\omega,\eta_0^{+\omega})} \text{ "if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\infty}(\mathcal{P}), y \text{ codes } (\mathcal{M}_\infty(\mathcal{Q},\Lambda)|\eta_0^{+\omega},\xi) \text{ then } (x,y) \in \tau_C\text{"}.$$

Now to show ($\Leftarrow$), let $(\mathcal{S},\Xi) \in I(\mathcal{Q},\Lambda)$ be such that for some $\gamma < \gamma^{\mathcal{S},0}_{D_i}$, $\xi = i^{\mathcal{S},\Xi}_{\mathcal{S},\infty}(\gamma)$. Let $\nu = i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\delta_0)$, we have

$$\mathcal{S} \vDash\!\Vdash_{Col(\omega,\nu^{+\omega})} \text{ "if } x \text{ codes } i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\mathcal{P}), y \text{ codes } (\mathcal{S}|\nu^{+\omega},\gamma) \text{ then } (x,y) \in i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\tau^{\mathcal{Q},\Lambda}_C)\text{"}.$$

This means there is a quasi-strategy Ψ on $\mathcal{S}(0)$ ($\mathcal{S}(0) = \mathcal{S}|(\nu^{+\omega})^{\mathcal{S}}$) such that $(\mathcal{S}(0), i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\Sigma))$ is 1-suitable, Ψ extends $i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\Sigma))$, and Ψ is D_i-iterable. We need to see that $\pi^{(\mathcal{S}(0),i^{\mathcal{Q},\Lambda}_{\mathcal{Q},\mathcal{S}}(\Sigma)),\infty}_{D_i}(\gamma) = \xi$. But this is true by the choice of D_i, $\xi = i^{\mathcal{S},\Xi}_{S,\infty}(\gamma)$, and the fact that Ψ agrees with Ξ on how ordinals below $\gamma^{\mathcal{S},0}_{D_i}$ are mapped.

The equivalence above shows $A \in \mathcal{M}_\infty(Q,\Lambda)$, hence completes the proof of (†). □

(†) in turns shows that η_0 is a cardinal in HOD and $\eta_0 \leq \Theta$ (otherwise, $V^{\mathrm{HOD}}_{\eta_0} = |\mathcal{M}_\infty(Q,\Lambda)|\eta_0| \vDash \Theta$ is not Woodin while HOD $\vDash \Theta$ is Woodin).

Claim 3. $\eta_0 \geq \Theta$.

Proof. Suppose toward a contradiction that $\eta_0 < \Theta$. Let $\mathcal{Q}(0) = \mathcal{Q}|(\delta_0^{+\omega})^{\mathcal{Q}}$, $\Lambda_0 = \Lambda|\mathcal{Q}(0)$, and $\mathcal{M}_\infty(\mathcal{Q},\Lambda)(0) = \mathcal{M}_\infty(\mathcal{Q},\Lambda)|(\eta_0^{+\omega})^{\mathcal{M}_\infty(\mathcal{Q},\Lambda)}$. Let $\pi = i \restriction \mathcal{Q}(0)$; so π is according to Λ_0. By the Coding Lemma and our assumption that $\eta_0 < \Theta$, $\pi, \mathcal{M}_\infty(Q,\Lambda)(0) \in N$. From this, we can show $\Lambda_0 \in N$ by the following computation: $\Lambda_0(\vec{\mathcal{T}}) = b$ if and only if

1. the part of $\vec{\mathcal{T}}$ based on P is according to Σ;

2. if $i^{\vec{\mathcal{T}}}_b$ exists then there is a $\sigma : \mathcal{M}^{\vec{\mathcal{T}}}_b \to \mathcal{M}_\infty(Q,\Lambda)(0)$ such that $\pi = \sigma \circ i^{\vec{\mathcal{T}}}_b$;

3. $\vec{\mathcal{T}}^\frown\mathcal{M}_b^{\vec{\mathcal{T}}}$ is Q-structure guided.

By branch condensation of Λ_0, (1),(2), and (3) indeed define Λ_0 in N. This means Λ_0 is OD^N from Σ (and some real x); hence $\Lambda_0 \in N$. So suppose $\gamma = w(Code(\Lambda_0)) < \Theta^\Omega$. In N, let

$$\begin{aligned} B = \{(\mathcal{R}, \Psi, x, y) \quad | \quad & (\mathcal{R}, \Psi) \text{ is a hod pair equivalent to } (\mathcal{P}, \Sigma),\ x \text{ codes } \mathcal{R},\ y \in A_\mathcal{R} \\ & \text{where } A_\mathcal{R} \text{ is the least } OD(Code(\Psi)) \text{ set such that } w(A_\mathcal{R}) > \gamma\} \end{aligned}$$

Then $B \in \mathbb{B}(\mathcal{P}, \Sigma)^N$. We may assume Λ_0 respects B. It is then easy to see that whenever $(\mathcal{R}, \Lambda_\mathcal{R}) \in I(Q(0), \Lambda_0)$ (also let $\mathcal{S} \lhd \mathcal{R}$ be the iterate of $\mathcal{P}$), $w(Code(\Lambda_\mathcal{R})) \geq w(A_\mathcal{R})$ because $\Lambda_\mathcal{R}$ can compute membership of $A_\mathcal{R}$ by performing genericity iterations (above $\mathcal{S}$) to make reals generic. This means $w(Code(\Lambda_\mathcal{R})) > \gamma = w(Code(\Lambda_0))$. This contradicts the fact that $w(Code(\Lambda_\mathcal{R})) = w(Code(\Lambda_0))$. □

Claim 3 implies then that $\eta_0 = \Theta$. Therefore, claims 2 and 3 give a contradiction to our initial assumption in N. This completes the proof of the lemma. □

Now we define a strategy Σ_∞ for $\mathcal{M}_\infty$ extending the strategy Σ_∞^- of $\mathcal{M}_\infty^- = \text{HOD}|\theta_\alpha$. Let $(\mathcal{P}, \Sigma, A) \in \mathcal{F}$ and suppose $\mathcal{P}$ is Σ-n-suitable with $\langle \delta_i \mid i < n \rangle$ being the sequence of Woodins of $\mathcal{P}$ above $\mathcal{P}^-$, let $\tau_{A,k}^{\mathcal{M}_\infty}$ = common value of $\pi_{\vec{B},\infty}^{\mathcal{P},\Sigma}(\tau_{A,\delta_k}^{\mathcal{P}})$. Σ_∞ will be defined (in V) for trees on $\mathcal{M}_\infty|\eta_0$ in $\mathcal{M}_\infty$. For $k \geq n$, $\mathcal{M}_\infty \vDash$ "$Col(\omega, \eta_n) \times Col(\omega, \eta_k) \Vdash (\tau_{A,n}^{\mathcal{M}_\infty})_g = (\tau_{A,k}^{M_\infty})_h \cap \mathcal{M}_\infty[g]$" where g is $Col(\omega, \eta_n)$ generic and h is $Col(\omega, \eta_k)$ generic and $(\tau_{A,n}^{\mathcal{M}_\infty})_g$ is understood to be $A_{(\mathcal{M}_\infty^-, \Sigma_\infty^-)} \cap \mathcal{M}_\infty[g]$. This is just saying that the terms cohere with one another.

Let $\lambda^{\mathcal{M}_\infty} = \sup_{i<\omega} \eta_i$. Let G be $Col(\omega, \lambda^{\mathcal{M}_\infty})$ generic over $\mathcal{M}_\infty$. Then let $\mathbb{R}_G^* = \bigcup_{i<\omega} \mathbb{R}^{\mathcal{M}_\infty[G\restriction\eta_i]}$ be the symmetric reals and $A_G^* := \bigcup_k (\tau_{A,k}^{\mathcal{M}_\infty})_{G|\eta_k}$.

Proposition 4.6. *For all $A \in \mathbb{B}(\mathcal{M}_\infty^-, \Sigma_\infty^-)$, $L(A_G^*, \mathbb{R}_G^*) \vDash \mathsf{AD}^+$*

Proof. We briefly sketch the proof of this since the techniques involved have been fully spelled out in the proof of previous lemmas. If not, using Σ_1-reflection, we obtain a model N (of a sufficient fragment of $\mathsf{ZF} + \mathsf{DC}$) coded by a Suslin co-Suslin set such that in N, $V = \text{Lp}^\Sigma(\mathbb{R})$ and the statement of the Proposition fails. Next we get a pair $(\mathcal{Q}, \Lambda)$ as in the proof of Theorem 4.2. The direct limit $\mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ extends $\mathcal{M}_\infty^N$. By the discussion in Section 2.2, a Λ-iterate of $\mathcal{Q}$ realizes N as its new derived model.

Working in N, let $A \subseteq \mathbb{B}(\mathcal{M}_\infty^-, \Sigma_\infty^-)$ be the least OD set such that $L(A_G^*, \mathbb{R}_G^*) \nvDash \mathsf{AD}^+$. Then there is an iterate $\mathcal{M}$ of $\mathcal{Q}$ having preimages of all the terms $\tau_{A,k}^{\mathcal{M}_\infty}$ for $k < \omega$. Now further iterate $\mathcal{M}$ to $\mathcal{M}'$ such that $\mathcal{M}'$ realizes N as its derived model; $\mathcal{M}'$ exists by the discussion in Section 2.2. By Theorem 2.12, $\mathcal{M}'$ thinks that its derived model satisfies that

$L(A_{(\mathcal{P},\Sigma)},\mathbb{R}) \vDash \mathsf{AD}^+$, where we reuse $(\mathcal{P},\Sigma)$ for an equivalent (but possibly different) hod pair from the original one; by elementarity, $\mathcal{M}$ thinks the same about the set of reals interpreted by $\langle \tau^{\mathcal{M}}_{A,k} \mid k<\omega\rangle$ in its derived model. Now iterate $\mathcal{M}$ to $\mathcal{M}_\infty(\mathcal{Q},\Lambda)$. By elementarity and the fact that $\mathcal{M}_\infty(\mathcal{Q},\Lambda)$ extends $\mathcal{M}^N_\infty$, $L(A^*_G,\mathbb{R}^*_G) \vDash \mathsf{AD}^+$. This is a contradiction. $\square$

Definition 4.7. *Given a normal tree $\mathcal{T} \in \mathcal{M}_\infty$ and $\mathcal{T}$ is based on $\mathcal{M}_\infty|\theta_0$. $\mathcal{T}$ is by Σ_∞ if the following hold (the definition is similar for finite stacks):*

- *If $\mathcal{T}$ is short then Σ picks the branch guided by the Q-structure (as computed in $\mathcal{M}_\infty$).*
- *If $\mathcal{T}$ is maximal then $\Sigma_\infty(\mathcal{T})$ = the unique cofinal branch b which moves $\tau^{\mathcal{M}_\infty}_{A,0}$ correctly for all $A \in OD$ such that there is some $(\mathcal{P},\Sigma,A) \in \mathcal{F}$ i.e. for each such A, $i_b(\tau^{\mathcal{M}_\infty}_{A,0}) = \tau^{\mathcal{M}^{\mathcal{T}}_b}_{A^*,0}$.*

Lemma 4.8. *Given any such $\mathcal{T}$ as above, $\Sigma_\infty(\mathcal{T})$ exists.*

Proof sketch. Suppose not. Again reflect the failure to a model N coded by a Suslin co-Suslin set. We may assume $N \vDash V = \mathrm{Lp}^\Sigma(\mathbb{R})$ where $(\mathcal{P},\Sigma)$ is a hod pair giving us $\mathrm{HOD}|\theta_\alpha$. Just as in the previous proposition, we then get a next mouse $\mathcal{Q}$ with strategy Λ such that $\mathcal{M}^N_\infty \trianglelefteq \mathcal{M}_\infty(\mathcal{Q},\Lambda)$. Λ has the property that for all $A \in \mathbb{B}(P,\Sigma)$, there is a Λ-iterate $(\mathcal{M},\Lambda_{\mathcal{M}})$ of $\mathcal{Q}$ such that $\Lambda_{\mathcal{M}}$ strongly respects A (see [3, Section 3.1] for the definition). This easily gives us a contradiction. $\square$

It is evident that $L[\mathcal{M}_\infty][\Sigma_\infty] \subseteq \mathrm{HOD}$. Next, we show $\mathcal{M}_\infty$ and Σ_∞ capture all unbounded subsets of Θ in HOD. In $L[\mathcal{M}_\infty][\Sigma_\infty]$, first construct (using Σ_∞) a mouse $\mathcal{M}^+_\infty$ extending $\mathcal{M}_\infty$ such that $\mathrm{o}(\mathcal{M}_\infty)$ is the largest cardinal of $\mathcal{M}^+_\infty$ as follows:

1. Let $\mathbb{R}^*_G$ be the symmetric reals obtained from a generic G over $\mathcal{M}_\infty$ of $\mathrm{Col}(\omega,<\lambda^{\mathcal{M}_\infty})$.
2. For each A^*_G (defined as above) (we know $L(\mathbb{R}^*_G,A^*_G) \vDash \mathsf{AD}^+$), S-translate the hybrid mice over $\mathbb{R}^*_G$ in this model to hybrid mice $\mathcal{S}$ extending $\mathcal{M}_\infty$. Let $\mathcal{S}_A$ be the union of such $\mathcal{S}$, then $D^+(\mathcal{S}_A,\lambda^{\mathcal{M}_\infty}) = L(\mathbb{R}^*_G,A^*_G)$.
3. Let $\mathcal{M}^+_\infty = \bigcup_A \mathcal{S}_A$.

By a Σ_1-reflection argument as above and the proof of [5, Theorem 0.1], we get [17]

(a) The translated mice over $\mathcal{M}_\infty$ are all compatible, don't project across $o(\mathcal{M}_\infty)$; hence 3. above makes sense.

[17]Suppose one of $(a),(b),(c)$ fails. Using Σ_1-reflection, let $N,\mathcal{Q},\Lambda,\Sigma,\mathcal{M}_\infty(\mathcal{Q},\Lambda)$ be as in Lemma 4.2. Let $\mathcal{Q}'$ be a Λ-iterate of $\mathcal{Q}$ such that $\mathcal{Q}'$ realizes N as its derived model and Σ' be the corresponding tail of Σ. It's first order over $\mathcal{Q}'$ that $\mathcal{Q}'$ is the union of all S-translations of Σ'-mice over $\mathbb{R}$ in $L(A,\mathbb{R})$ for $\mathrm{Code}(\Sigma') <_w A$ and A is OD in N. By elementarity, in N, $\mathcal{M}^+_\infty = \mathcal{M}_\infty(\mathcal{Q},\Lambda)$. Then $(a)-(c)$ easily follows. Contradiction.

(b) $\mathcal{M}^+_\infty$ is independent of G; in particular, $\mathcal{M}^+_\infty \in V$.

(c) $\mathcal{M}^+_\infty$ contains as its initial segments all translation of $\mathbb{R}^*_G$-mice in $D^+(\mathcal{M}^+_\infty, \lambda^{\mathcal{M}_\infty})$.

(a)-(c) intuitively gives us that $\mathcal{M}^+_\infty$ contains enough mice to compute HOD. We remark that $\mathcal{M}^+_\infty = \mathcal{M}_\infty$ in the analysis of HOD in $L(\mathbb{R})$ (cf. [9]). In general, though, the two premice could be distinct.

The following is the key lemma.

Lemma 4.9. *HOD* $\subseteq L[\mathcal{M}_\infty][\Sigma_\infty]$

Proof. Using Theorem 2.4, we know HOD $= L[P]$ for some $P \subseteq \Theta$. Therefore, it is enough to show $P \in L[\mathcal{M}_\infty][\Sigma_\infty]$. Let ϕ be a formula defining P, i.e.

$$\alpha \in P \Leftrightarrow V \vDash \phi[\alpha].$$

We suppress the ordinal parameter here. Now in $L(\mathcal{M}_\infty, \Sigma_\infty)$ let

$$\pi : \mathcal{M}_\infty|(\eta_0^{++})^{\mathcal{M}_\infty} \to (\mathcal{M}_\infty)^{D^+(\mathcal{M}^+_\infty, \lambda^{\mathcal{M}_\infty})}$$

where π is according to Σ_∞.

Claim. $\alpha \in P \Leftrightarrow D^+(\mathcal{M}^+_\infty, \lambda^{M_\infty}) \vDash \phi[\pi(\alpha)]$. $\quad$ (†)

Proof. Otherwise, Σ_1-reflect the failure of (†) as before to get a model N coded by a Suslin co-Suslin set, a hod pair $(\mathcal{P}, \Sigma)$ giving us HOD$|\theta_\alpha$ such that

$$N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AD}^+ + V = \mathrm{Lp}^\Sigma(\mathbb{R}) + (\exists \alpha)(\phi[\alpha] \not\Leftrightarrow D^+(\mathcal{M}^+_\infty, \Sigma_\infty) \vDash \phi[\pi(\alpha)]).$$

Fix such an α. We may assume N is minimal with the above property. As before, let $\mathcal{Q}, \Lambda$ be as in the proof of Theorem 4.2. We may assume Λ is guided by $\vec{D}$ where $\vec{D} = \langle D_n \mid n < \omega \rangle$ is defined as in Lemma 4.5. It's easy to see that $\mathcal{M}^+_\infty = \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$. Let $\sigma : \mathcal{Q}|((\delta_0^{\mathcal{Q}})^{++})^{\mathcal{Q}} \to (\mathcal{M}_\infty)^{D^+(\mathcal{Q}, \lambda^{\mathcal{Q}})}$ be the direct limit map by Λ. By replacing $\mathcal{Q}$ by a Λ-iterate far enough into the direct limit system given by Λ if necessary, we may assume $\sigma(\overline{\alpha}) = \alpha$ for some $\overline{\alpha}$. Working in N, it then remains to see that:

$$D^+(\mathcal{M}^+_\infty, \lambda^{\mathcal{M}_\infty}) \vDash \phi[\pi(\alpha)] \Leftrightarrow D^+(\mathcal{Q}, \lambda^{\mathcal{Q}}) \vDash \phi[\sigma(\overline{\alpha})]. \quad (**)$$

To see that $(**)$ holds, first let $i : \mathcal{Q} \to \mathcal{Q}'$ be according to Λ and $\mathcal{Q}'$ realizes N as its derived model. We need to see that the fragment of $\Lambda_{\mathcal{Q}'}$ that defines $i(\sigma(\overline{\alpha}))$ can be defined in $D^+(\mathcal{Q}', \lambda^{\mathcal{Q}'})$. This then will give the equivalence in $(**)$. Because $\alpha < \eta_0$, $\overline{\alpha} < \delta_0$, pick an n such that such that $\gamma^{\mathcal{Q},0}_{D_n,0} > \overline{\alpha}$ so

$$\gamma^{\mathcal{Q}',0}_{D_n,0} > i(\overline{\alpha}).$$

Let

$$\sigma' : \mathcal{Q}'|((\delta_0^{\mathcal{Q}'})^{++})^{\mathcal{Q}'} \to (\mathcal{M}_\infty)^{D^+(\mathcal{Q}',\lambda^{\mathcal{Q}'})}.$$

be the direct limit map given by $\Lambda_{\mathcal{Q}'}$.

Then the fragment of $\Lambda_{\mathcal{Q}'}$ that defines $\sigma'(i(\overline{\alpha}))$ is definable from D_n (and $\mathcal{Q}'|(\delta_0^{\mathcal{Q}'})$) in $D^+(\mathcal{Q}',\lambda^{\mathcal{Q}'})$ and in fact $i(\sigma(\overline{\alpha})) = \sigma'(i(\overline{\alpha}))$. By elementarity, $(**)$ follows.

The equivalence $(**)$ gives us a contradiction. $\square$

The claim clearly implies $P \in L[\mathcal{M}_\infty][\Sigma_\infty]$. $\square$

Lemma 4.9 implies HOD $= L[\mathcal{M}_\infty][\Sigma_\infty]$, hence completes the proof of Theorem 1.3.

Proof of Corollary 1.4. The case Θ being limit in the Solovay sequence is really a consequence of [3]. Recall that by [3], HOD$|\Theta \vDash \mathsf{GCH}$. In the case Θ is a successor in the Solovay sequence, the proof of Theorem 1.3 shows that HOD$|\Theta \vDash \mathsf{GCH}$.

Now Woodin has shown that if $\mathsf{AD}^+ + V = L(\wp(\mathbb{R}))$ holds, then HOD $= L[P]$ for some $P \subseteq \Theta$ (see [16] for a proof). This means

$$\forall\alpha \geq \Theta \text{ HOD} \vDash 2^\alpha = \alpha^+.$$

This completes our proof of Corollary 1.4. $\square$

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