## HOMEWORK 6 ANSWER KEYS

**Problem 1(a):** The statement is FALSE. We shall prove this by contradiction. We suppose (for contradiction) that "There are only finitely many even Fibonacci numbers". Let $a_1, a_2, \ldots, a_k$ be those numbers (we list them in increasing magnitude, so $a_k$ is the largest even Fibonacci number). This means: for any $n$, if $F_n > a_k$, then $F_n$ is odd. Fix an $n$ such that $F_n > a_k$. So both $F_n$ and $F_{n+1}$ are odd. But $F_{n+2} = F_n + F_{n+1}$ is then even (being a sum of two odd numbers). This contradict the fact that $F_{n+2} > a_k$.

**Problem 1(b):** We prove this by (strong) induction. Here, the property $P(n)$ is "$F_n \leq 2^n$".

Base case: $n = 1 \Rightarrow F_n = 1 \leq 2^1 = 2$. $n = 2 \Rightarrow F_n = 1 \leq 2^2 = 4$.

Inductive step: We prove: $\forall n \geq 1\ P(1) \wedge P(2) \cdots \wedge P(n) \wedge P(n+1) \Rightarrow P(n+2)$.

So fix $n \geq 1$ and assume $P(1), P(2), \ldots, P(n), P(n+1)$ hold. We need to show $P(n+2)$ holds, i.e. $F_{n+2} \leq 2^{n+2}$. We know: $F_{n+2} = F_n + F_{n+1}$. Using the inductive hypothesis, we know $F_n \leq 2^n$ and $F_{n+1} \leq 2^{n+1}$. Hence

$$F_{n+2} \leq 2^n + 2^{n+1} \leq 2^{n+1} + 2^{n+1} = 2^{n+2},$$

as desired.

**Problem 2:** Let $P(n)$ be the statement "7 divides $2^{n+2} + 3^{2n+1}$".

We want to show "$\forall n \in \mathbb{N}\ P(n)$" by minimum counterexample. Suppose the statement fails, i.e.

$$C = \{k \mid 7 \text{ divides } 2^{k+1} + 3^{2k+1}\} \neq \emptyset.$$

Let $m = min(C)$. Note that $m \geq 1$ because 7 divides $2^{0+2} + 3^{2*0+1} = 7$. So we can write $m = k + 1$ for $k \geq 0$. By the definition of $m$, $k \notin C$ (i.e. 7 divides $2^{k+2} + 3^{2k+1}$).

So $2^{m+2} + 3^{2m+1} = 2^{k+3} + 3^{2k+3} = 2 * 2^{k+2} + 9 * 3^{2k+1} = 2 * (2^{k+2} + 3^{2k+1}) + 7 * 3^{2k+1}$.

By the assumption on $k$, 7 divides $2*(2^{k+2}+3^{2k+1})$. Obviously, 7 divides $7*3^{2k+1}$. So 7 divides their sum, which is $2^{m+2} + 3^{2m+1}$. This contradicts the assumption that $m \in C$.

I leave this as an easy exercise to do the above argument using induction. All of the main ideas are presented in the proof above; you just have to put them together.

**Problem 5.4.1:** We prove this by strong induction. $P(n)$ is the statement "3 divides $b_n$".

Base case: $n = 1$: $b_n = 3$ by definition and clearly 3 divides $b_1$.

$n = 2$: $b_n = 6$ by definition and 3 divides 6.

Inductive step: We show: $\forall n \in \mathbb{N}\ P(1) \wedge P(2) \wedge \cdots \wedge P(n) \wedge P(n+1) \Rightarrow P(n+2)$.

Fix $n \in \mathbb{N}$ and assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \wedge P(n+1)$ is TRUE. We show $P(n+2)$ is TRUE. This means we want to show 3 divides $b_{n+2}$. But $b_{n+2} = b_n + b_{n+1}$. By the inductive hypothesis, 3 divides $b_n$ (this is $P(n)$) and 3 divides $b_{n+1}$ (this is $P(n+1)$). So 3 divides $b_n + b_{n+1} = b_{n+2}$ as desired.

**Problem 5.3.2:** (a) The proof's calculations are correct. The only incorrect thing is it treats

$P(n)$ as if it were a number. The proof writes: "$P(n) = \Sigma_{k=0}^{n} r^k$" (similarly, "$P(0) = \ldots$", "P(n+1) $= \ldots$"). Keep in mind that $P(n)$ is a proposition (or statement) about $n$ and $P(n)$ can either be TRUE or FALSE, and it is not a number; so do not write "$P(n) = \ldots$".
(b) The proof is as given, you just have to rewrite, e.g. the first line to "Let $P(n)$ be "$\Sigma_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$"". And replace "$P(0) = \ldots$" with "$P(0)$ is $\Sigma_{k=0}^{0} r^k = \frac{1-r^{0+1}}{1-r}$, which is true" (similarly for $P(n+1)$).