DESCRIPTIVE INNER MODEL THEORY, LARGE CARDINALS, AND COMBINATORICS
RESEARCH STATEMENT

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My research interest is in mathematical logic and set theory. My current research focuses on studying the connections between inner models, (determined) sets of reals, hybrid structures (such as HOD[1] of determinacy models), forcing, and strong combinatorial principles (such as the Proper Forcing Axiom (PFA), (generalizations of) the tree property, the Unique Branch Hypothesis (UBH)). I'm also interested in applications of strong forcing axioms such as PFA and their connection with other combinatorial principles in set theory.

The research described below mostly belongs to an area of set theory called *descriptive inner model theory* (DIMT). DIMT is an emerging field in set theory that explores deep connections between *descriptive set theory* (DST) and *inner model theory* (IMT). DST studies a certain class of well-behaved subsets of the reals and of Polish spaces (e.g. Borel sets, analytic sets) and has its roots in classical analysis, through work of Baire, Borel, Lebesgue, Lusin, Suslin and others. One way a collection Γ of subsets of a Polish space can be well-behaved is that they satisfy various regularity properties, e.g. they have the Baire property, every uncountable set in Γ contains a perfect subset, every set in Γ is Lebesgue measurable. A cornerstone in the history of the subject is the discovery of the Axiom of Determinacy (AD) by Mycielsky and Steinhaus in 1962. AD states that every infinite-length, two-person game of perfect information where players take turns play integers is determined, i.e. one of the players has a winning strategy. If every set in $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is determined, then they have all the regularity properties listed above (and more), and hence very well-behaved. AD contradicts the axiom of choice as the latter implies the existence of very irregular sets like the Vitali set; however, inside a universe of ZFC, there may be many interesting sub-universes (models) that satisfy AD, for instance, $L(\mathbb{R})$ the minimal transitive class model of ZF that contains the reals may satisfy AD. One important and fruitful branch of descriptive set theory studies structure theory of models of AD. IMT forms one of the core subjects in modern set theory; its main objective is study "canonical" models of various extensions of ZFC, called *large cardinal axioms* (or simply large cardinals) and construct such models under various circumstances (e.g. see question (2) below). The large cardinal axioms form a linear hierarchy of axioms (in terms of consistency strength) extending ZFC and every known, natural axiom in mathematics/set theory is decided by one such axiom.[2] The first "canonical model" of large cardinals is *Gödel's constructible universe* L, the minimal model of ZFC. It is well-known that L cannot admit "very large" large cardinals; *the Gödel's inner model program*, a major program in inner model theory, aims to construct and analyze L-like models that can accommodate larger large cardinals under various hypotheses. Benchmark properties that help determine the canonicity of these models include *the Generalized Continuum Hypothesis* (GCH), Jensen's $\square$-principles (see Question (1), more details later).

[1] HOD stands for the class of "Hereditarily Ordinal Definable" sets, see [7] for a definition.

[2] By Gödel's incompleteness theorem, given any axiom (A) extending Peano Arithmetic, one can construct a sentence, albeit unnatural, which is not decidable by (A).

DIMT uses tools from both DST and IMT to study and deepen the connection between canonical models of large cardinals and canonical models of AD. One of the first significant developments in DIMT comes in the 1980's with works of Martin, Steel, Woodin and others; their work, for instance, shows that one can construct models of AD (e.g. they showed AD holds in $L(\mathbb{R})$) assuming large cardinal axioms (those that involve the crucial notion of Woodin cardinals) and conversely, one can recover models of large cardinal axioms from models of AD. The key to uncover these connections is to analyze structure theory of models of AD. Much of DIMT and my research go along this line (e.g. see Question (1) below). The conjecture that "PFA has the exact consistency strength as that of a supercompact cardinal" is one of the most longstanding and arguably important open problems in set theory. Techniques recently developed in DIMT and to some extent from the research described here also enable us to make significant progress in calibrating the consistency strength lower-bound for PFA (Question (2), to be discussed in detail later).

We now describe these connections in more technical details in the next couple of paragraphs.[3] One way of formalizing this connection is through *the Mouse Set Conjecture* (MSC), which states that, assuming the AD or a more technical version of it (AD^+), then whenever a real x is ordinal definable from a real y, then x belongs to a canonical model of large cardinal (mouse) over y. MSC conjectures that the most complicated form of definability can be captured by canonical structures of large cardinals. An early instance of this is well-known theorem of Shoenfield that every Δ^1_2 real is in the Gödel's constructible universe L. Another instance is a theorem of W.H. Woodin's that in the minimal class model containing all the reals, $L(\mathbb{R})$, if AD holds, then MSC holds. However, the full MSC is open and is one of the main open problems in DIMT. Instances of MSC have been proved in determinacy models constructed by their sets of reals (much larger than $L(\mathbb{R})$) and these proofs typically obtain canonical models of large cardinals (mice) that capture the relevant ordinal definable real by analyzing HOD of the determinacy models. Hence, the key link between these two kinds of structures (models of large cardinals and models of determinacy) is the HOD of the determinacy models. A central notion in the proof of MSC and the analysis of HOD is the notion of *hod mice* (developed by G. Sargsyan, cf. [19], which built on and generalized earlier unpublished work of H.W. Woodin), which features heavily in my research described here. Hod mice are a type of models constructed from an extender sequence and a sequence of iteration strategies of its own initial segments. The extender sequence allows hod mice to satisfy some large cardinal theory and the strategies allow them to generate models of determinacy. Unlike pure extender models, there are many basic structural questions that are still open for hod mice (to be discussed later).

Another way of exploring the determinacy/large cardinal connection is via the Core Model Induction (CMI), which draws strength from natural theories such as PFA to inductively construct canonical models of determinacy and those of large cardinals in a locked-step process by combining core model techniques (for constructing the core model K) with descriptive set theory, in particular the scales analysis in $L(\mathbb{R})$ and its generalizations. CMI is the only known systematic method for computing lower-bound consistency strength of strong theories extending ZFC and it is hoped that it will allow one to compute the exact strength of important theories such as PFA. CMI is another central theme of my research. Much work has been done the last several years in developing techniques for CMI and it's only recently realized that constructing hod mice in non-AD^+ contexts (e.g. in the context of PFA, in contrast to Sargsyan's constructions) is a key step in CMI.

A large part of my research described here is devoted to studying structural properties of hod mice, developing methods for constructing hod mice in non-AD^+ contexts, and applying these constructions in the core model induction. One of the main goals is to construct determinacy models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular", "$\mathsf{AD}^+ +$ the largest Suslin cardinal is on the Solovay sequence"

[3] Non-logicians may want to simply skim this through.

(LSA) and beyond, from various theories such as PFA, the existence of countably closed guessing models (cf. [44]) (see Section 3), and failures of *the Unique Branch Hypothesis* (UBH) (see Section 3). These lower-bounds are beyond the reach of pure core model methods.

There are many problems in DIMT that I've been interested in and working on but my hope is to make progress toward answering two fundamental questions in the area (to be discussed in details later):

(1) Is HOD of a determinacy model fine-structural (e.g. do the $\mathsf{GCH}, \square$ hold in HOD)?

(2) What is the consistency strength of PFA?

The three main aspects of my research are as follows: (a) connections between inner models, hybrid structures, and canonical sets of reals; (b) applications of the structure theory of the three hierarchies in (a) and their connections; (c) strong combinatorial principles, determinacy, and large cardinals through the lens of forcing. These three areas will be described in the last three sections. I will give more background for each area and discuss selected problems that will generate interesting results and progress in each area. Many of the important problems listed below are either (indirectly) related to or (directly) elaborated from problems (1) and (2) above and hence quite ambitious. However, I think these problems are worth pursuing as they are important for advancing the field. I have obtained results related to most of them in the past.

Section 1 discusses some selected relevant papers that serve as background for the topics discussed above. The last 3 sections discuss aspects (a), (b), and (c) respectively of my research. Discussions related to questions (1) and (2) are done at various points in Sections 1, 2, 3. The casual reader can skip Section 1 and return to it when necessary. If one is interested in the work done regarding questions (1) and (2), one can simply read Sections 2 and 3. Other results and applications of the basic work described Sections 2 and 3 are discussed in Section 4.

1. RECENT RELATED RESULTS AND PAPERS

This section summarizes the content of published and recently completed papers that are relevant to my research. These papers give some context to the research described in the last three sections. The reader can skip this section and come back when needed.

(i) Trang, N., HOD in natural models of AD^+, Annals of Pure and Applied Logic, 165(10), 2014, 1533-1556

(ii) Trang, N., PFA and guessing models, to appear in Israel Journal of Mathematics, 49 pages

(iii) Schlutzenberg, F. and Trang, N., Scales in hybrid mice over $\mathbb{R}$, arXiv:1210.7258v4, 95 pages

(iv) Schlutzenberg, F. and Trang, N., The fine structure of operator mice, arXiv:1604.00083v2, 46 pages

(v) Trang, N., Determinacy in $L(\mathbb{R}, \mu)$, Journal of Mathematical Logic, 14(01), 2014, 23 pages

(vi) Trang, N., Structure theory of $L(\mathbb{R}, \mu)$ and its applications, Journal of Symbolic Logic, 80(01), 2015, 29-55.

(vii) Sargsyan, G. and Trang, N., Non-tame mice from tame failures of the unique branch hypothesis, Canadian Journal of Mathematics, 66(4), 2014, 903-923

(viii) Sargsyan, G. and Trang, N., Tame failures of the unique branch hypothesis and models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular, submitted to the Journal of Mathematical Logic, 28 pages

(ix) Trang, N., Derived models and supercompact measures on $\wp_{\omega_1}(\wp(\mathbb{R}))$, Mathematical Logic Quarterly, 61(1-2), 2015, 56-65

Paper (i), built on many years of work by J.R. Steel, H.W. Woodin, G. Sargsyan amongst others (cf. [19], [40]), completes the characterization of HOD in a large class of determinacy models. These models satisfy theories as strong as "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular". The paper contributes to the progress towards answering question (1) above; in particular, it shows that HOD of these determinacy models is fine structural and hence satisfies, for instance, $\mathsf{GCH}, \forall\kappa\ \square_\kappa$ and $\diamondsuit_\kappa$. Furthermore, the methods of computing HOD in the paper, which go back to work of Steel and Woodin and generalizations by Sargsyan, show that HOD can be construed as the limit of a certain direct limit system of iterable hybrid extender models; it turns out that this type of ideas can be used in many other situations, one of which is in computing lower-bound consistency strength of various theories (cf. [44], [20], [24], [21], [43]).

Paper (ii), inspired by previous work of Steel [34] and Sargsyan [23], shows roughly that PFA is a very strong principle in the sense that it has very high lower-bound consistency strength. In particular, the paper shows the existence of inner models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular" from a variety of combinatorical principles, some of which follow from PFA (e.g. threadability, a strengthening of failures of squares) and some of which aren't even consistent relative to PFA (e.g. the existence of ω_2-guessing models, a strengthening of the tree property at ω_3). The paper makes significant progress toward the solution of question (2) above; in particular, it improves significantly the lower-bound consistency strength of PFA, whose previously known lower-bound has been established in [8] a decade ago. The main technique employed in the paper, which is quite different from that used in [8], is the core model induction method. This method, pioneered by Woodin, turns out to have wide-ranging and far-reaching consequences (to be discussed more later). I am very hopeful that generalizations of the methods used in this paper can be used to push lower-bound strength of PFA (and other combinatorial principles) further up.

Papers (iii) and (iv) develop the basic theory of a certain class of hybrid extender models and generalize Steel's scales analysis [35] and [36] to that in hybrid extender models over $\mathbb{R}$. In particular, the papers shows that if an operator F has nice condensation properties then one can develop fine structure theory of F-mice (structures constructed from an extender sequence and F) and construct iterable F-mice with large cardinals from appropriate hypotheses (via a relativized backgrounded construction, called $K^{c,F}$-constructions). Furthermore, paper (iv) proves that the scales propagation theorems hold in F-mice over $\mathbb{R}$ under optimal hypotheses. This work, whose results expand our basic understanding of hybrid structures, is crucial in applications involving the core model induction (cf. paper (ii) [44], [24], [20], and [45]).[4]

Paper (v), whose main theorem is discovered independently by Woodin and me, gives an optimal characterization of determinacy in minimal models of "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact". It solves a conjecture by Woodin and shows that in such models, if one knows "$\Theta > \omega_2$", which is a very weak consequence of the Moschovakis's Coding Lemma and hence of determinacy, one gets full determinacy. The paper contributes to the general area of set theory which investigates consequences of compactness principles on ω_1 under $\mathsf{ZF} + \mathsf{DC}$ and structural properties of models of these types of theories. It also illustrates the versatility of the core model induction method

[4] The standard reference for the core model induction [26] develops a version of the theory of F-mice. The theory developed in paper (iii) [29] is quite different from that in [26] though its original motivation is to fix some of the mistakes that occur in [26].

by showing that core model induction can be used, besides computing lower-bound consistency strength, to prove theorems on structural properties of various class of models (a precursor to the core model induction method was used by Woodin to prove the equivalence of determinacy and Turing determinacy in $L(\mathbb{R})$). A consequence of the main result of paper (v) is that the theory "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact" is equiconsistent with "ZFC + there are ω^2 many Woodin cardinals"; this uses results and techniques (due to Woodin) in paper (vi)).

Paper (vi) investigates structural properties of minimal determinacy models of "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact" and give some applications. The measure witnessing ω_1 is $\mathbb{R}$-supercompact is called *the Solovay measure*; this type of measures was first shown to exist in models of $\mathsf{AD}_{\mathbb{R}}$, the axiom of determinacy for infinite games on reals, by Solovay [33]. It is worth mentioning that the theory "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact" equiconsistent with "ZFC + there is a measurable cardinal" and hence very weak (see paper (v) [48] for a proof). Paper (v) shows that "$\mathsf{AD} + \mathsf{ZF} + \mathsf{DC} + \omega_1$ is $\mathbb{R}$-supercompact" is equiconsistent with "ZFC + there are ω^2 many Woodin cardinals", and hence much stronger; though this is known to be logically and consistently weaker than $\mathsf{AD}_{\mathbb{R}}$. The paper also proves uniqueness of the Solovay measure in these minimal models and shows that the theory "$\mathsf{AD} + \omega_1$ is $\mathbb{R}$-supercompact". These two results, due to Woodin, use important and indispensible techniques of descriptive inner model theory, including choiceless Prikry forcings and derived model constructions. As applications, in paper (vi), using $\mathbb{P}_{\max}$-forcing over such models, I construct an ideal on $\mathcal{P}_{\omega_1}(\omega_2)$, the collection of countable subsets of ω_2, with very high homogeneity property and computes its exact consistency strength. This type of ideals is of interest since it also satisfies $\mathsf{WRP}^*_2(\omega_2)$, a combinatorial principle isolated by Woodin in [53]. I also give a full analysis of HOD in the paper, providing a positive answer for Question (1) in such models. The methods used in this paper are also useful in investigating models of more general compactness properties on ω_1, for instance models of generalized Solovay measures [42] as well as answering global questions about these types of models (cf. Conjecture 3.7).

Papers (vii) and (viii) concern the strength of certain failures of Unique Branch Hypothesis (UBH). UBH, a fundamental hypothesis in set theory isolated by Martin and Steel [11], states that every iteration tree on V has at most one cofinal well-founded branch. UBH has been shown to have failed for a certain class of iteration trees by work of Neeman and Steel [17] and Woodin [54]. However, their counterexamples are not nice, in that the trees being constructed are either overlapping or extenders chosen are not suffiently closed. UBH for nice trees is crucial in constructing canonical inner models of large cardinals (e.g. of superstrong, supercompact, and beyond), the main objective of inner model theory. Determining whether UBH holds for nice trees is, therefore, a fundamental problem in set theory. Papers (vii) and (viii) show that it is very hard for UBH for nice trees to fail by showing that the failure of UBH for nice trees implies the existence of models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. The papers seem to suggest that one should try to prove that UBH for nice trees holds. These improve significantly earlier results of Steel [38]. The papers contribute new methods for constructing complicated, iterable models of large cardinals; previously, these types of models were only constructed, in one way or another, using covering type arguments (cf. [34], [20]), which heavily depend on whether the background universe satisfies some form of failures of squares. These papers accomplish this without such assumptions and hence I hope the techniques developed here will have applications in future projects.

Paper (ix) concerns derived models. Derived models (at a limit λ of Woodin cardinals) are maximal models of determinacy in the symmetric extension of $Col(\omega, <\lambda)$. Derived model constructions are the only known method for constructing canonical models of determinacy (i.e. models of the form $V = L(\mathcal{P}(\mathbb{R}))$) from large cardinals. Determinacy models of the form $L(\mathcal{P}(\mathbb{R}))[X]$ (those that are constructed from their sets of reals and a predicate for some class X) have gained attention recently; some of them turn out to be quite crucial (for instance, models that papers (v) and (vi) deal with are of this form). Paper (ix) shows that under suitable large cardinal assumption it is possible to show that if λ is a limit of Woodin cardinals and M is the derived model at λ, then $M[\mu] \cap \mathcal{P}(\mathbb{R}) = M \cap \mathcal{P}(\mathbb{R})$ where μ is the club filter on $\mathcal{P}_{\omega_1}(\mathcal{P}(\mathbb{R}))$. The point is μ is not in M but the model obtained by constructing from M and μ does not generate new sets of reals, and hence $M[\mu]$ necessarily satisfies determinacy. The methods used in the paper are important in constructing "expansions" of derived models and investigating their properties. Woodin, unpublished, has shown that it is consistent (relative to large cardinals) that ω_1 is supercompact (and $\mathsf{ZF} + \mathsf{DC}$ holds); though it is not clear that his model satisfies AD. I hope the methods used in the paper will generalize to tackle the problem of constructing a model of determinacy and ω_1 is supercompact.

2. INNER MODELS, HYBRID STRUCTURES, AND CANONICAL SETS OF REALS

From now on, instead of AD, we actually will talk about AD^+, a technical completion of AD isolated by Woodin in the 1980's. The reader will lose nothing by pretending that AD is AD^+. It is a conjecture that the two principles are equivalent and this is supported by the fact that every known model of AD that has ever been constructed satisfies AD^+. Under AD (or AD^+), we define Θ as the supremum of ordinals α for which there is a surjection from $\mathbb{R}$ onto α. The Solovay sequence $(\theta_\alpha : \alpha \leq \Omega)$ associated to an AD^+ model is a club subset of Θ with $\theta_\Omega = \Theta$ and the θ_α's are defined similarly to Θ but with a definability restriction on the surjections. Roughly speaking, the longer the Solovay sequence is, the stronger the associated AD^+ theory is. Also, $\mathsf{AD}_\mathbb{R}$ models all have Solovay sequence of limit length.

The problem of analyzing HOD of AD^+ models has gradually grown into a central problem of inner model theory and spurs the development of descriptive inner model theory. One main reason for its importance is that it provides insights into the relationship between canonical inner models of large cardinals (pure extender models) and models of AD^+ (as alluded to earlier). Unlike HOD of ZFC models which are more or less intractable from the point of view of inner model theory, HOD of AD^+ models in some sense are very well-behaved and code up the AD^+ models in a canonical way; hence understanding HOD of such models provide more insights into the models themselves. It turns out that HOD of an AD^+ model (at least up to model of theories that we have been able to understand and analyze) is a strategic extender model, a kind of hybrid structure that is constructible from a sequence of extenders and iteration strategies of its own initial segments. In other words, HOD contains large cardinal information (coded by its extender sequence) as well as information about the determinacy world (coded by its iteration strategy of its own initial segments). The picture going forward is that one should think of HOD of AD^+ models as a bridge that connects large cardinal universes and determinacy worlds. The three hierarchies: the pure extender models, the AD^+ models, and their HOD (the strategic extender models) are therefore intimately connected. The main goal of descriptive inner model theory is to understand this interconnectedness. This kind of understanding has been giving rise to many types of applications. For example, techniques from the HOD analysis allow one to calibrate the exact consistency strength of determinacy theories by constructing inner models of large cardinals that are in some sense have the same information as the corresponding determinacy model. Another source of applications is through the core model induction, which relies heavily on our understanding of HOD of AD^+ models (see the next section for a more detailed discussion of this topic).

Question 2.1. *Is HOD of an* AD^+ *model fine-structural? In particular, does HOD satisfy* GCH*,* $\forall\kappa\ \square_\kappa$ *and* $\diamondsuit_\kappa$ *hold?*

This is question (1) above and I regard this as one of the central questions to tackle in his long-term research plan. The conjecture is that the answer to the question is positive. The first breakthrough in answering Question 2.1 is by work of Steel and Woodin in the 1990's [40] for $L(\mathbb{R})$, the minimal model of AD^+ (if there are models of AD^+). In particular, Steel shows that HOD up to Θ in $L(\mathbb{R})$ is a pure extender model of large cardinals and Woodin, building on Steel's work, shows that full HOD of $L(\mathbb{R})$ is a strategic extender model (so HOD knows a fragment of its own iteration strategy). The results and techniques in [40] provide a template for analyzing HOD of bigger models of AD^+. For instance in 2009, Sargsyan [19] gives an analysis of HOD (up to Θ) of all AD^+ models up to $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. After results of Steel and Woodin, [19] is regarded as a landmark in the field as it provides many useful techniques for constructing hod mice (strategic extender models that generate initial segments of HOD). In paper (i) [46], I complete the full HOD analysis and answer positively Question 2.1, for AD^+ models up to $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular.

There has been progress in answering Question 2.1 for AD^+ models beyond $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. In particular, a very strong determinacy theory that is at the forefront of the subject and has been studied extensively by various authors, including me, in recent years is called LSA or "$\mathsf{AD}^+ + \Theta = \theta_{\alpha+1} + \theta_\alpha$ is the largest Suslin cardinal". LSA was isolated by Woodin in [53] but has not been known to be consistent until very recently. The general structural theory of LSA as well as the HOD analysis for models of such theories is the subject of the upcoming book by G. Sargsyan, and me [22]. The book proves the consistency of LSA and answers Question 2.1 positively by showing that HOD of models of LSA is a strategic extender model of a certain kind and so $HOD \vDash \mathsf{GCH}$. However, the structure theory of HOD here is so much more complicated than those in AD^+ models up to $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular that it's not clear other combinatorial principles like $\Box, \diamondsuit$ hold in HOD. In particular, it is still open whether HOD in LSA models satisfy $\forall\kappa\ \diamondsuit_\kappa$. However, I show in [22] that HOD satisfies $\forall\kappa\ \Box_{\kappa,2}$[5] by combining the techniques developed in [22] for hod mice and the construction of $\Box$ by Schimmerling-Zeman in [25]. The solution, as a result, is not strictly inner model theoretic; it carries some descriptive set theoretic flavor and this seems essential since HOD carries with it information about determinacy world after all. This result is important for various types of applications which are also discussed in the book, one of which is to prove the consistency of LSA from PFA (to be discussed in more details in the next section, cf. Question 3.1).

It should be mentioned that the AD^+ models discussed above are all of the form $V = L(\mathcal{P}(\mathbb{R}))$. In paper (v) [48], I give the full HOD analysis in AD^+ models of the form $L(\mathcal{P}(\mathbb{R}))[\mu]$ where μ is a Solovay measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$. Here the HOD analysis requires new ingredients since HOD is influenced not only by the sets of reals in the model but also by μ. The analysis combines techniques for constructing direct limit systems (by Steel and Woodin in [40]) and a variation of Prikry forcings using the measure μ. The upshot is [48] answers Question 2.1 positively for this class of models and this analysis plays crucial role in analyzing global questions regarding these models (to be discussed in the next section, cf. Conjecture 3.7).

The non-uniformity of the hierarchy developed in [22] makes it very difficult to prove full $\Box$ holds in HOD; the hierarchy in [22] is too extender-biased. One way to tackle Question 2.1 is to redesign/redefine HOD or more generally, the notion of hod mice to make it non-extender-biased. Such a work has been done by John Steel in [41]. The advantage of the hierarchy defined in [41] is that we no longer need to "layer"; instead, strategies and extenders are fed into the models in a uniform way. Using the results in [41], Steel and I recently showed able to answer Question 2.1 for AD^+ models much stronger than LSA. More precisely, we show that

Theorem 2.2 (AD^+). *Assume $\mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathcal{P}(\mathbb{R}))$ and no mice with long extenders exist. Assume Hod Pair Capturing holds. Then in HOD, for all κ: κ is not subcompact if and only if $\Box_\kappa$ holds.*

Since the structures of hod mice in [41] are similar to those of extender models; the proof of Theorem 2.2 naturally relies on the proof of $\Box$ in extender models (cf. [25]). Two new ingredients

[5] $\Box_{\kappa,2}$ is a slightly weaker combinatorial principle than $\Box_\kappa$.

of the proof are: the use of the $\mathfrak{B}$-operator in [29] to feed branch information of various trategies into a hod premouse and a method of phalanx comparisons (similar to that used in [41]) used to prove a key condensation lemma.

In order to carry out the HOD analysis, generally, one has to prove structural theorems about the AD^+ models. The following are two central problems in descriptive inner model theory that one needs to address first before answering Question 2.1 (for any particular AD^+ theory that one is studying). Recent development in the theory of hod mice (cf. [41]) suggests that the answer to Question 2.1 can be answered in full generality assuming variations of 2.3, 2.4.

Question 2.3 (Generation of pointclasses)**.** *Given an AD^+ model M, suppose $\Gamma \subsetneq \mathcal{P}(\mathbb{R})^M$ is a strict Wadge initial segment of $\mathcal{P}(\mathbb{R})^M$. Is it possible to generate Γ by an iterable (hybrid) model?*

Conjecture 2.4 (Mouse Set Conjecture)**.** *Assume no mice with superstrong cardinals exist and AD^+ holds. Suppose $x, y \in \mathbb{R}$ are such that x is ordinal definable from y (i.e. $x \in OD(y)$). Then there is a mouse $\mathcal{M}$ over y such that $x \in \mathcal{M}$.*

Quesion 2.3 asks whether strategic extender iterable structures (like hod mice) can generate complicated pointclasses of AD^+. This is crucial for the HOD analysis since these iterable structures are used in direct limit systems that generate initial segments of HOD; roughly speaking, the more complicated the pointclass generated by the structure, the longer the initial segment of HOD can be captured by the corresponding direct limit system). The positive answer to Quesion 2.3 is given up to LSA by (in chronological order): Steel and Woodin [40] in L($\mathbb{R}$), by Woodin for a broader class of models [39], by Sargsyan [19] for a larger class still (up to minimal models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular), and finally by [22] for all AD^+ models up to minimal models LSA.

Conjecture 2.4, isolated by Steel and Woodin in the 1990's after their landmark results about HOD in $L(\mathbb{R})$, roughly states that the most complicated form of definability (ordinal definability) is captured by iterable structures (mice). A strengthening of it relativizes ordinal definability to ordinal definability relative to a strategy Σ of a hod mouse; we call this Strong Mouse Set Conjecture (SMSC). The proof of MSC (and SMSC) is particularly important and its main point is: it shows a tail of a hod pair $(\mathcal{P}, \Sigma)$ is captured by a pure-extender model. This idea is key in computing the exact consistency strength of a determinacy theory and has far-reaching consequences in other applications. SMSC has been solved for AD^+ models up to LSA in [39], [40], [19], and [22].

To tackle Questions 2.3, and 2.4 for AD^+ theories beyond LSA, it seems that we need to understand hod mice with certain properties and how to construct them[6]. We don't seem to have a good understanding of such objects (but see [41]); this topic is therefore at the forefront of descriptive inner model theory and is central to my long-term research plan.

To end this section, we list one more question which is important for various applications. An operator F is *nice* if it has properties resembling mouse operators that come from the AD^+ world, like condensation, relativizing well, and determines itself on generic extensions.

Problem 2.5. *Given a nice operator F, develops fine-structure theory of hybrid extender models of the form $L[\mathbb{E}, F]$, prove that iterable $L[\mathbb{E}, F]$ models with large cardinals exist (we call these F-mice), and analyze the scales pattern in F-mice over $\mathbb{R}$.*

Problem 2.5 is an integral part of developing the theory of hybrid mice suitable for core model induction applications. Partial answers to Problem 2.5 are given in paper (iv) [29]. The answers in [29] are partial in the sense that the authors assume additionally that F, in some sense, is fine-structural and has some smallness properties. This is enough for applications that aim to

[6] i.e. those with extenders overlapping Woodin cardinals.

construct models up to LSA (from large cardinals or strong combinatorial principles) but seems hardly complete. There are various non-fine-structural operators that are of interest. Schlutzenberg and I, in paper (iv) [28], subsequently found a way to develop the theory of F-mice without assuming F is fine-structural by carefully formulating the notion of hybrid premouse and F's condensation properties[7]. Though the theory developed there is not general enough to cover some important non-fine-structural operators, like the C_Γ operator. We will discuss applications of Problem 2.5 and the theory developed in [28], and [29] in the next section.

3. CORE MODEL INDUCTIONS, FORCING AXIOMS, AND COMPACTNESS PRINCIPLES

The main sort of applications of the material discussed previously is in calibrating the consistency strength of set-theoretic principles. The main tool for accomplishing this is the core model induction. The core model induction, the only known systematic approach for computing lower-bound consistency strength, is pioneered by Woodin and developed further by Steel, Schindler and others to construct models of AD^+ from strong set-theoretic principles (cf. [26], [34], [2], [37]). The methods of these papers typically show AD^+ holds in $L(\mathbb{R})$ and manage to construct models of slightly stronger AD^+ theories. They are, however, insufficient to construct models of $\mathsf{AD}_\mathbb{R} + \Theta$ is regular, LSA and beyond because of the lack of understanding of stronger AD^+ models, F-mice for complicated operators F, and the scales analysis in F-mice over $\mathbb{R}$. There are many ingredients that need to be put together to accomplish this task, some of which are: understanding HOD of complicated AD^+ models, i.e. those with "long" Solovay sequence (e.g. see [19], [46], [22] by Sargsyan, Steel, and me), developing a general enough theory of F-mice and various techniques for constructing new scales and pointclasses of determinacy (cf. T. Wilson [51], F. Schlutzenberg and me [29]).

We discuss below several problems involving determining consistency strength of important set-theoretic principles. The arguably most important one is Question (2) above.

Question 3.1. *What is the consistency strength of* PFA*?*

As mentioned above, [34], [23], the paper (i) [44], and the book [22] show that the strength of PFA is at least that of LSA, which is a very strong determinacy principle.

Theorem 3.2 (Sargsyan-Trang)**.** *Assume* PFA*. Then there are inner models M such that* $M \vDash \mathsf{LSA}$*.*

This is the strongest lower-bound consistency result for PFA. However, LSA is shown to be consistent relative to a Woodin limit of Woodin cardinals (cf. [22]) and the consensus amongst set theorists is that PFA should be as strong as a supercompact cardinal (it is well-known that the upper bound for PFA is a supercompact, cf [1]). A complete answer to Question 3.1 is the holy grail of inner model theory. From the point of view of descriptive inner model theory (this is the view I take), to completely solve this problem, one needs to understand HOD of AD^+ models; in particular, one needs to resolve Questions 2.1, 2.3 and generalizations of Conjecture 2.4. Futhermore, at the level of $\mathsf{AD}_\mathbb{R} + \Theta$ is regular and beyond, one needs to do a significant amount of work to construct hod-like objects to eventually generate HOD of the AD^+ models; this is where the bulk of the construction is and seems to be hypothesis-dependent. For example, [22], combining techniques for constructing K^c from core model theory and techniques for analyzing HOD of AD^+ models,

[7] The condensation properties in [26] are inadequate for developing the theory of F-mice as pointed out in paper (iii).

introduces a strategic K^c construction, whose outcome is a hod-like object $\mathcal{M}$ that generates HOD of an LSA model (and hence the LSA model itself). The existence of $\mathcal{M}$ is established using covering arguments taking advantage of PFA and my theorem (mentioned above) that $\mathcal{M} \vDash \forall\kappa\, \square_{\kappa,2}$.

One short-coming of the construction in [22] is the fact that we are unable to prove that $\mathcal{M}$ is iterable (we get around this by showing some definable hull of $\mathcal{M}$ is iterable and this suffices to get a model of LSA). However, for going beyond LSA the iterability of $\mathcal{M}$ seems important. I wish to pursue this further as I think the methods introduced by [22] has a lot of mileage in further advancing the solution of Question 3.1.

Earlier this year, I observed that

Theorem 3.3. *Assume* PFA+ *there is a Woodin cardinal. Then there are inner models that satisfy* ZFC+ *there is a Woodin cardinal which is a limit of Woodin cardinals.*

So with an extra, mild large cardinal assumption, we can improve the lower-bound of Theorem 3.2 significantly. Unfortunately, the method of Theorem 3.3 does not seem to generalize. So it seems to me the right approach to systematically studying the universe of sets and its canonical structures in the presence of strong forcing axioms like PFA is continue to generalize the approach of [22]

We now discuss other combinatorial principles which are important in their own rights. The following question samples four important theories. Below, guessing models are the strongest known generalizations of the tree property. The existence of κ-guessing models implies the tree property at κ^+. ω_1-guessing models were first introduced by Viale and Weiss [50], and the obvious formulations of κ-guessing models (for $\kappa > \omega_1$) are introduced in [49] and [44]. And $\mathsf{MM}(\kappa)$ is the Martin Maximum for posets of size at most κ for a cardinal κ.

Question 3.4. *(a) What is the consistency strength of $\neg\square_\kappa$ for some singular strong limit κ?*

(b) What is the consistency strength of $\neg\square_\kappa + \neg\square(\kappa)$ for a regular $\kappa \geq \aleph_3$ such that $\kappa^\omega = \kappa$ and $2^{<\kappa} = \kappa$?[8]

(c) What is the consistency strength of $\mathsf{MM}(\mathsf{c}^+)$*?*

(d) What is the consistency strength of the existence of ω_1-guessing models?

The constructions in [44] and [22] actually can be used to construct models of LSA from the following principles:

(I) GCH + there is a cardinal κ such that κ is countably closed and for all $\alpha \in [\kappa, \kappa^{+4}]$, $\neg\square(\alpha)$;

(II) there exist stationary many ω_2-guessing models that are countably closed;

(III) there exists a strongly compact cardinal.

I hope that these methods can be applied to the situations in Question 3.4. More precisely, since (a), (b), (c) above are local principles, there is less room to work with so the exact methods used above don't seem to work here. However, I expect that appropriate refinements of these methods can be used to make some progress, at least in constructing models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular from (a), (b), and (c). Regarding (d), note that ω_1-guessing models are not countably closed (unlike the situation for ω_2-guessing models), so some of the covering arguments used in [44] don't work for

[8] $\square(\alpha)$ says that there is a sequence of $(C_\beta : \beta < \alpha)$ such that C_β is a club subset of β, $C_\beta \cap \gamma = C_\gamma$ for every limit point γ of C_β, and there is no "thread" through the sequence, i.e. there is no club C in α such that for any limit point β of C, $C \cap \beta = C_\beta$.

(d). However, I hope that frequent extension techniques and covering arguments without countable closure in [13] can be adapted to tackle (d). To the best of my knowledge, (b), (c), and (d) have lower-bound roughly that of $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ (by [8]) and (a) has lower-bound $\mathsf{AD}_{\mathbb{R}}$ (by [34] and [23]).

Conjecture 3.5. *The consistency strength of* $\mathsf{MM}(\mathsf{c})$ *is exactly that of* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is regular.*

Woodin in [53], using $\mathbb{P}_{\max}$ forcing techniques shows that $\mathsf{MM}(\mathsf{c})$ holds in a $\mathbb{P}_{\max}$-generic extension of any model of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. Neeman and Schimmerling [16], using a different method, also force $\mathsf{MM}(\mathsf{c})$ from a large cardinal property much weaker than supercompact. $\mathsf{MM}(\mathsf{c})$ is of interest by set theorists since it implies various combinatorial principles at H_{ω_2}, for instance, it decides the size of the continuum, the powerset of ω_1 (they both equal $\aleph_2$), it implies the nonstationary ideal on ω_1 is saturated and the weak reflection principle $\mathsf{WRP}_2(\omega_2)$. Steel and Zoble [37] show that $\mathsf{MM}(\mathsf{c})$ implies AD holds in $L(\mathbb{R})$ and this result is one of the first core model induction arguments there was. I expect that the knowledge from recent work can be used to improve the lower-bound of $\mathsf{MM}(\mathsf{c})$. The advantage here is that we now understand models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular very well, much better than when Steel and Zoble proved their result; furthermore, more techniques have been discovered in constructing models of determinacy since.

Question 3.6. *Show that the failure of* UBH *for nice trees imply the existence of models of* LSA.

As mentioned, UBH for nice trees (normal, non-overlapping, and extenders are sufficiently closed in the models they are chosen from) is central in constructing canonical inner models of large cardinals (up to supercompact and beyond). Counterexamples of UBH for non-nice trees have been constructed by Neeman, Steel, and Woodin (discussed above). One way of enforcing the belief that UBH for nice trees is true is to show that the failure of the principle has very high consistency strength. Martin and Steel [11] took the first step along this direction by showing that the failure of UBH implies the existence of a Woodin cardinal (in some inner model). Steel [38] improves this to $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ (and a bit beyond this in some cases). The methods Steel uses are traditional core model theoretic methods. Sargsyan and me in [21] and [24], using the core model induction, improve upon Steel's results significantly and show that the failure of UBH for nice trees implies the existence of models of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular. As mentioned above, the methods developed in [21] and [24] are different from those developed in [44] and [22] etc. since we cannot use covering-type arguments in this case; in particular, it is not obvious that the strategic K^c construction introduced above converges in this situation (after all, the theorem that $\mathcal{M}$ satisfies $\forall\kappa\ \square_{\kappa,2}$ does not seem useful here). New methods for constructing models of LSA are needed here.

The next four questions concern compactness principles on ω_1 in the context of $\mathsf{ZF} + \mathsf{DC}$, a topic that has been important to me since his student days at UC Berkeley and plays a major part in his thesis [42]. The first one addresses the uniqueness problem for canonical models of $\mathsf{AD}^+ + \omega_1$ is $\mathbb{R}$-supercompact (i.e. AD^+ models of the form $L(\mathbb{R})[\mu]$).

Conjecture 3.7. *Assume* AD *or* ZFC. *Then there is at most one* AD^+*-model of the form* $L(\mathbb{R})[\mu]$ *where* μ *is the Solovay measure on* $\mathcal{P}_{\omega_1}(\mathbb{R})$ *in the model.*

This conjecture grows out of a question asked by Woodin [52] in the 1980's. The question asks whether assuming determinacy, there must be at most one canonical model of $\mathsf{AD}^+ + \omega_1$ is $\mathbb{R}$-supercompact. Woodin in [52] shows that assuming determinacy, if ω_1 is $\mathbb{R}$-supercompact then there must be a unique measure witnessing this. So the question about the uniqueness of models is natural and resembles the situation regarding uniqueness of minimal models of one measurable cardinal (which Kunen gives a positive answer to).

The question had been open for more than 30 years until very recently. In a joint work with D. Rodríguez [18], we answer the question positively in the AD case and Rodríguez, by refining the techniques in [18], has settled the conjecture fully. The main techniques grow out of the my HOD analysis in [48]; the main point is that the HOD analysis allows us to compare and line up hod mice in different AD^+-models of the form $L(\mathbb{R})[\mu]$ and conclude that on a cone of reals x, HOD_x of the models are the same. This implies the model must be the same via Vopenka-like forcing methods developed by Woodin. In [18], we also make significant progress towards settling Conjecture 3.7 by showing the conjecture is true if one assumes additionally a very mild large cardinal property. Without the additional large cardinal assumption, it seems one needs to understand better the universally Baire sets in V and develops some methods for constructing iterable models with large cardinals in the case that there are distinct such models.

Under $\mathsf{AD}_{\mathbb{R}}$, there is a class $(\mu_\alpha : \alpha < \omega_1)$ of (generalized Solovay) measures where μ_α concentrates on sequences of length ω^α of elements in $\mathcal{P}_{\omega_1}(\mathbb{R})$; here μ_0 is the original Solovay measure. These measures were constructed by Solovay (for $\alpha = 0$) in [33] and by Martin and Woodin independently (for $\alpha > 0$). The original proof uses determinacy for long game on reals of length α to construct μ_α. [42] gives a different construction (due to Woodin) of these measures and investigates the exact consistency strength of the theories $\mathsf{AD}^+ + \mu_\alpha$ exists. Conjecture 3.7 can be generalized to canonical models of generalized Solovay measures. [42] also proves determinacy transfer theorems (from Π^1_1 to $<$-ω^2-Π^1_1) for real games of length α (for each $\alpha < \omega_1$) by analyzing HOD of determinacy models of the form $L(\mathbb{R})[\mu_\alpha]$ and using techniques from [14].

[42] also gives a construction of the measure μ_∞ (called the ultimate measure) which concentrates on countable sequences of elements in $\mathcal{P}_{\omega_1}(\mathbb{R})$ and computes the exact consistency strength of such measures. The interesting thing here is that μ_∞'s existence does not follow from determinacy of long games that are provable from $\mathsf{AD}_{\mathbb{R}}$ (unlike that of the μ_α's) and hence there is no analogous determinacy transfer theorem for μ_∞ like the above (hence the question below). To make progress toward answering the question, one needs to analyze HOD of AD^+ models of the form $L(\mathbb{R})[\mu_\infty]$ and convert the iterability of structures that come out of the HOD analysis to determinacy of long games.

Question 3.8. *What is the relationship between the ultimate measure μ_∞ and long game determinacy?*

Moving on to higher forms of compactness, in [45], the T. Wilson and I investigate various forms of compactness on ω_1 (beyond $\mathbb{R}$-strong compactness and supercompactness). The main results of [45] show that for a fixed set X, the principles "ω_1 is X-strongly compact" and "ω_1 is X-supercompact" are generally not equiconsistent (this is certainly the case for $X = \mathcal{P}(\mathbb{R})$ as shown in [45]) but they seem to interleave in consistency strength. The paper [45] develops techniques for carrying out the core model induction in contexts where the full axiom of choice fails; these techniques along with the HOD analysis in [19] and [22] can hopefully shed new lights on the full answers to the following question, Question 3.9. The techniques developed in [45] also give unexpected and surprising results. For example, one of the consequences of methods from [45] is a result of T. Wilson's and mine that under a certain smallness assumption, $\mathsf{AD}_{\mathbb{R}}$ is equivalent to Blackwell-$\mathsf{AD}_{\mathbb{R}}$. This is a higher analogue of similar results of Martin and Neeman regarding Blackwell-AD [12].

Question 3.9. *What is the exact consistency strength of ω_1 is strongly compact (supercompact)? Is "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ strong compact" equiconsistent with "$\mathsf{ZF} + \mathsf{DC} + \omega_1$ supercompact"?*

We should remark that the theory ω_1 is supercompact (hence strongly compact) is consistent relative to a proper class of Woodin limits of Woodins as shown by Woodin (unpublished); this bound seems to suggest that solutions to Questions 3.9 and 3.10 are within the scope of current inner model theory. The model built by Woodin is by putting club filters on various sets of the form $\mathcal{P}_{\omega_1}(\lambda^\omega)$ on top of the Chang model and is of interest in its own right (see [22, Chater 11] for similar types of models). This kind of ideas combined with ideas from [47] hopefully allow us to make progress toward answering

Question 3.10. *Build a model of* AD^+ *and* ω_1 *is supercompact.*

The following two questions concern two fairly different topics and are joint work with M. Zeman. The first concerns Woodin's theorem on Σ^2_1-absoluteness conditioned on CH. [9].

Question 3.11. *1. What is the exact consistency strength of* Σ^2_1*-absoluteness conditioned on* CH*?*

2. What is the exact consistency strength of $\Sigma^2_1(uB)$*-absoluteness conditioned on* CH*?*

Woodin has shown (cf. [9]) that Σ^2_1-absoluteness conditioned on CH is consistent relative to a measurable Woodin cardinal. An easy observation from his proof shows that if one additionally assumes there is a proper class of Woodin cardinals, one gets the consistency of Question 3.11(2), here we allow for universally Baire sets as parameters. Zeman and I initial result is that we show a lowerbound for both of these principles is AD holds in $L(\mathbb{R})$ and the second principle enables us to go farther. The upper-bounds seem to be a bit beyond the scope of current inner model theory but our lower-bound proof seems to suggest that the upper-bound can be reduced to below that of a measurable Woodin. We also conjecture that the second principle is strictly stronger than the first.

Question 3.12. *Analyze the extent of self-iterability of tame, iterable extender models and beyond.*

Question 3.12 concerns self-iterability of a sufficiently iterable fine structural extender model, and consequently, ordinal definability inside such a model. This is a task remotely resembling HOD computations in determinacy models. The first systematic result along these lines is due to Schindler-Steel [27] where they proved that certain levels of a tame extender model have iteration strategies inside the model. The existence of such iteration strategies allows to define levels of the model without referring to the extender sequence. The goal in this project is to analyze the class of iteration trees with respect to which the model is internally iterable, as well as perform ordinal definability computations; such computations naturally rely on self-iterability results. The focus here are models beyond tame. A natural example of such a model is $\mathcal{M}^\sharp_{adr}$, the extender model which corresponds to the consistency strength of $\mathsf{AD}_\mathbb{R}$. This question comes from [39], and there are several indications that the study of self-iterability of such models may be relevant for studying their derived models. Zeman and I realize that methods developed in [30] for different purposes has a strong potential to be modified to yield a variety of self-iterability results. We have settled, using the aforementioned methods, the self-iterability problem at the level of bland extender models (in the sense of [15]) for iterable universal extender models. This extends results in [27] for the class of universal extender models. The methods employed combined with recent results of F. Schlutzenberg also give complete answers to self-iterability of minimal models of a fixed theory, like $\mathcal{M}^\sharp_{adr}$.

4. COMBINATORICS, FORCING, AND LARGE CARDINALS

This section samples some of the problems regarding forcing the consistency of some combinatorial principles of interest above. The first question regards guessing models.

Question 4.1. *(a) Is the theory "for every $1 < n < \omega$, for every $1 \leq m \leq n$, there are stationary many ω_m-guessing models" consistent?*

(b) Is the theory "for every $1 \leq n < \omega$, there are stationary many ω_n-guessing models" consistent (relative to ω supercompact cardinals)?

Recall guessing models are the strongest generalizations of the tree property. In [50], it is shown that the existence of ω_1-guessing models is a consequence of PFA. For a fixed $n > 1$, ω_n-guessing models do not follow from PFA; however, I [44] show that the existence of these models hold in some generic extension of V if V has a supercompact cardinal. I asked, during my talk at the American Institute of Mathematics (Palo Alto) in 2014, Question 4.1. I. Neeman later told me that he can force models of part (a) using his forcing with models of several types machinery. Part (b) appears to be open. I hope that techniques by Cummings-Foreman from [4] and Fontanella from [6], which roughly interleave collapse forcings with Mitchell-type forcings that force the tree property, can be used to tackle the question.

Question 4.2 seems reasonable in light of the result regarding ω_1-guessing models in [50]. The answer seems to hinge on better understanding of recent development in higher analogues of PFA (e.g. Neeman's $\{\omega_1, \omega_2\}$-PFA).

Question 4.2. *What is the relationship between ω_2-guessing models with higher analogues of* PFA*?*

The next question was motivated by Conjecture 3.7. It is natural to consider the situation for models of the form $L(\mathbb{R})[\mu]$ that fail to satisfy AD^+ and ask whether these models are unique. D. Rodríguez has shown recently that the answer is "no" by forcing two distinct such models. Rodríguez assumes a measurable cardinal κ with Mitchell order κ^{++} (and no models with a Woodin cardinal exist) for his argument. The main technical point of the argument (which uses the smallness assumption) is the existence and local definability of the core model K.

Question 4.3. *Prove that the consistency strength of distinct non-AD^+ models of the form $L(\mathbb{R})[\mu]$, where μ is the Solovay measure in the model is that of a measurable cardinal.*

It is easy to see that if a model of the form $L(\mathbb{R})[\mu]$ exists, then there is a ZFC model with a measurable cardinal. For the converse, one appears to need class forcing techniques to construct two distinct measures μ_1, μ_2 of Mitchell order 0 on some κ and at the same time codes up definability of the ground model into a set-sized object that is in the models constructed by the reals and μ_1 and μ_2 respectively.

Question 4.4. *Prove that there does not exist a nice iteration tree $\mathcal{T}$ on V with two distinct cofinal branches b, c such that the limit models along b and c are the same.*

Question 4.4 provides insights into the central problem of proving UBH holds for nice trees. Steel has observed recently that if there is a nice iteration tree $\mathcal{T}$ on V as in the statement of Question 4.4, then there must be a model with a measurable Woodin cardinal; this suggests that it seems very difficult to construct such a $\mathcal{T}$. In fact, it is not even known how to construct a tree $\mathcal{T}$ as in Question 4.4 such that $\delta(\mathcal{T})$, the supremum of the critical points of extenders used in $\mathcal{T}$, is not in the range of the branch embeddings (if $\delta(\mathcal{T})$ is a fixed point of both branch embeddings, then [38] shows that the limit branch models must be distinct; this is the features of all counter-examples to UBH constructed so far). Schlutzenberg and I have observed that if $V = \mathrm{HOD}$ then there can be no such $\mathcal{T}$ as in Question 4.4. The strategy for solving Question 4.4 in general,therefore, is to perform a class forcing construction over V to force $V = \mathrm{HOD}$ and at the same time ensuring the forcing is closed enough that one can lift the branch embeddings to the generic extension, and finally one has to make sure that the resulting direct limit models along the branches continue to be the same.

The next topic involves developing descriptive set theory at the level of $V_{\lambda+1}$ for some singular cardinal λ (traditional descriptive set theory concerns about subsets of $V_{\omega+1}$). How forcing, combinatorics, and large cardinals come into play in the situation we have in mind will be explained below. Particularly, the following question is of special interest.

Question 4.5. *Assume there is a nontrivial elementary embedding j from $L(V_{\lambda+1})$ to $L(V_{\lambda+1})$ such that the critical point of j is below λ. Develop descriptive set theory for subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ and study structure theory of $L(V_{\lambda+1})$.*

The hypothesis in the question is called I_0 and is one of the strongest known large cardinal axioms not known to be inconsistent with ZFC. It should be mentioned that λ above is singular with cofinality ω. For other values of λ (e.g. inaccessible or with uncountable cofinality), descriptive set theory for $V_{\lambda+1}$ has been extensively studied by various mathematicians, most notably S.D. Friedman and his co-workers. Developing descriptive set theory for λ with cofinality ω faces many challenges. Without strong large cardinal axioms, such as I_0, in the background, it seems very difficult or even impossible to develop the basic theory that is useful.

I_0 was formulated by Woodin in the 1980's and was used to give one of the first proofs of the consistency of AD (he shows AD holds in $L(\mathbb{R})$ of the generic extension obtained by collapsing V_λ to ω); hence it seems reasonable to expect that there is some interesting descriptive set-theoretic structure for subsets of $V_{\lambda+1}$ under I_0 that resembles descriptive set theory for sets of reals under AD. This large cardinal property was further studied and developed by Laver [10] and Woodin [55]. Various authors, including X. Shi and I in [31], V. Dimonte and S.D. Friedman [5], using the structure theory of I_0 embeddings developed above and various forcing techniques, have studied combinatorial structures at λ^+, including showing that I_0 is consistent with the failure of SCH at λ (this was also claimed by Woodin). [31] proves this result using diagonal Prikry forcing and a generic absoluteness theorem proved by S. Cramer [3]. These methods and their variations are very useful for studying subsets of $V_{\lambda+1}$ under I_0.

Using reflection methods for I_0 embeddings and diagonal Prikry forcings, [3] and [32] have shown that under I_0, all subsets of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ have the perfect set property. Unfortunately, the same techniques don't seem to shed any light into the situation of the Baire property. Cramer and I observe that the obvious formulation of the Baire category theorem is false in this situation; this forces us to weaken the Baire category theorem and modify the Baire property accordingly. The main question is whether all sets in $L(V_{\lambda+1})$ have this (modified) Baire property. In fact, there is a plethora of quetions/projects that constitute parts of the Question 4.5. We list the ones that are of particular interest to me and will be part of his future research plans below (under I_0):

- Prove all sets in $L(V_{\lambda+1}) \cap \mathcal{P}(V_{\lambda+1})$ have the (modified) Baire property.
- Investigate the large cardinal structure for cardinals below Θ, the supremum of α such that there is a surjection from $V_{\lambda+1}$ onto α in $L(V_{\lambda+1})$. In particular, prove the ω-club filter on λ^+ is an ultrafilter (this is analogous to the fact that the club filter on ω_1 is an ultrafilter under AD).
- Investigate to what extent sets in $L(V_{\lambda+1}) \cap \mathcal{P}(V_{\lambda+1})$ have tree representations (analogous to Suslin representation for sets of reals).

Woodin has shown that λ^+ is a measurable cardinal in $L(V_{\lambda+1})$ but the measures he constructs concentrate on a stationary set. Cramer [3] has improved upon Woodin's result and shows that the weak ω-club filter is an ultrafilter on λ^+ and has made significant progress in analyzing tree representations in the context of I_0. Solving this question has other implications on the structure of λ^+, including whether every stationary subset of λ^+ reflects. Laver [10], Woodin [55], and Cramer have developed various notions of tree representations for subsets of $V_{\lambda+1}$ and prove propagation theorems about them. But these representations don't seem to imply regularity of subsets of $V_{\lambda+1}$ (like Baire property, uniformizations etc.).

Lastly, I discuss a recent joint project with D. Ikegami concerning forcings that preserve of the Axiom of Determinacy. The basic questions that we would like to tackle are

Question 4.6. *Assume AD^+. Classify what forcings $\mathbb{P}$ with the property that whenever $g \subseteq \mathbb{P}$ is V-generic:*

1. *there is an elementary embedding $j : V \to V[g]$ definable over $V[g]$;*
2. *$V[g] \vDash \mathsf{AD}^+$.*

Clearly, if (1) holds then (2) holds. Some partial answers have been known. For instance, we know that if $V = L(X)$ for some set X then (1) fails. The method for proving this, unfortunately does not work for general AD^+ models.

We hope to bring in tools discussed in Sections 1 and 2 into understanding the model $V[g]$. In particular, in the region where the HOD analysis holds, we hope to have a clearer picture of how the sets of reals in $V[g]$ are related to those in V. Some partial results we have obtained are as follows. First, using the Coding Lemma, we can show that if $\mathbb{P}$ adds a bounded subset of Θ^V but does not add any reals, then $V[g]$ does not satisfy AD^+. Using the HOD analysis, we show that below models of LSA, if $V \vDash \mathsf{AD}_{\mathbb{R}}$, then no $\mathbb{P}$ has the property that there is an elementary embedding $j : V \to V[g]$. Some technical improvements of these results have also been obtained.

Our conjecture regarding (1) above is that no $\mathbb{P}$ has the property that there is an elementary $j : V \to V[g]$. This would be the corresponding result to Kunen's famous theorem for ZFC models.

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