MATH 13 FINAL REVIEW AND SAMPLE PROBLEMS

1. BASIC LOGIC

Recall definitions of propositions, logical equivalence of propositions, basic connectives: $\vee, \wedge, \Rightarrow, \Leftrightarrow$, $\neg$ and their truth tables. Be comfortable with proving two propositions are logically equivalent using truth tables as well as using basic identities (like De Morgan's laws).

It's very important that you are comfortable with mathematical statements involving quantifiers $\exists, \forall$. You should practice deciding whether a statement is true or false. Practice translating statements in English, pseudo-mathematical into fully mathematical statements and negating such statements.

Be comfortable with writing proofs (direct proofs, proofs by cases, proofs by contrapositive, proofs by contradiction, proofs by counterexamples). Make sure to indicate which proof technique you are about to do in your proof; it helps the reader follow your logic more easily.

PROBLEMS: Go back and work on the homework problems and worksheet problems in discussion corresponding to each category above. Here are some additional problems if you feel like more practice.

Problem 1.1. Let P, Q, and R be propositions. Prove or disprove the following:

(a) $((P \wedge Q) \Rightarrow R)$ is logically equivalent to $((P \wedge (\neg R)) \Rightarrow (\neg Q))$.

(b) $((P \wedge Q) \Rightarrow R)$ is logically equivalent to $((Q \wedge (\neg R)) \Rightarrow (\neg P))$.

Problem 1.2. Decide whether the following statements are true or false. Write them in full mathematical form and negate them:

(a) If n is prime and $n > 2$ then n is odd.

(b) If m is even and n is odd, then $m + n$ is odd.

(c) If the function f is differentiable at a then f is continuous at a.

Problem 1.3. Prove/disprove the following statements:

(a) For every two distinct real numbers a and b, either $\frac{a+b}{2} > a$ or $\frac{a+b}{2} > b$.

(b) Let $n \in \mathbb{Z}$. Then $3n^3 + 4n^2 + 5$ is even if and only if n is even.

(c) For every odd positive integer n, $3|(n^2 - 1)$.

(d) There is no smallest positive real number.

(e) $\sqrt{3}$ and $\sqrt{2} + \sqrt{3}$ are irrational.

2. ARITHMETIC ON INTEGERS, MODULUS ARITHMETIC, AND INDUCTION

Recall the definitions of remainders, congruence, $\equiv \pmod{n}$, and the division algorithm.

Be comfortable with modular arithmetic (you should know some basic properties of addition, multiplication, exponentiation $\pmod{n}$ and applications of modular arithmetic in computing remainder of a number m when it is divided by n).

You should be able to compute $\gcd(a, b)$ using the Euclidean algorithm and be able to express $\gcd(a, b)$ as $ma + nb$ for some m, n. Finally, you should know the theorem which tells you how to find all integer points on a line $ax + by = c$ (Theorem 3.13 in the notes).

Be able to write a correct induction and/or strong induction proof and minimum counter-example proof. The set-up is very important; you should pay close attention to how you set up an induction proof.

For example, suppose you want to prove by strong induction the statement "for some nonnegative integer k, for all integers $n \geq k$, the statement $P(n)$ holds". You do the following.

Base step: We show $P(k)$ holds. (Depending on the problem, you may have one than one base case, e.g. you may have to check $P(k+1)$ (and/or $P(k+2)\dots$) holds. Recall, in the example of Fibnonacci sequence, we need to check $P(1), P(2)$ hold in the base case.)

Inductive step: We assume for some $n > k$, for all $k \leq m < n$, $P(m)$ holds. (This is our (strong) inductive hypothesis). We now show $P(n)$ holds.

PROBLEMS: Again, redo the relevant problems from the homework and discussion worksheets. Here are some additional problems on induction proofs (for the other topics, the homework problems suffice).

PROBLEM 2.1. Prove the following statements (using any method you want).

(a) $4^n > n^3$ for every positive integer n.

(b) $1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for every positive integer n.

PROBLEM 2.2. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 4$, $a_3 = 9$, and

$$a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3)$$

for all $n \geq 4$. Find a formula for a_n in terms of n and verify your formula is correct.

PROBLEM 2.3. Write a proof by strong induction and a proof by minimum counter-example of the following: that for each integer $n \geq 28$, there are nonnegative integers x, y such that $n = 5x+8y$. In fact, you should do this problem both ways: using strong induction and using minimum counter-example.

Remark 2.1. *The next two sections will be the main focus of the final (I estimate that about 65-70 % of the final will be on these topics).*

3. SET THEORY

Be comfortable with various ways of describing a set (by listing its elements, set-builder notations etc.). Recall definitions of $\subset$, $\subseteq$, $\cap$, $\cup$ (including $\bigcap, \bigcup$ of indexed collections of sets), $B \backslash A$, A^c as well as the negations of these operations. Recall definitions of relations, functions, Cartesian products, ordered pairs, powerset operation. You should practice on proving relations of sets using logic (e.g. $(A \cap B)^c = A^c \cup B^c$, $A \subseteq B$ and $C \subseteq D$ implies $A \times C \subseteq B \times D$, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$). Be able to compute unions and intersections of various collections of indexed sets.

Recall definitions of injections, surjections, bijections, domain, codomain, range of functions, inverse and compositions.

Important: You should be able to check if some f is a function, whether f is one-to-one, onto, or bijective and compute domain, range of f.

Review facts about finite sets (e.g. the proof that: for any finite sets A, B: $|A \times B| = |A|.|B|$, $|\mathcal{P}(A)| = 2^{|A|}$).

PROBLEMS: Redo as many problems from homework and worksheet as you can (these problems are important). Once you're comfortable with those problems you can practice the following.

PROBLEM 3.1. Do problem 6.3.4, 6.1.3.

PROBLEM 3.2. Show that the following function $f : \mathbb{N} \to \mathbb{Z}$ is a bijection: for each natural number n, let $f(n) = (-1)^n fl(n/2)$. The "floor" function fl is defined as follows. First for each real number x, let $fl(x)$ be the "floor of x", i.e. $fl(x)$ is the greatest integer less than or equal to x. For example, $fl(5.5) = 5$, $fl(-2.5) = -3$, $fl(3) = 3$.

PROBLEM 3.3. Compute the $\bigcup_{n \in \mathbb{N}, n \geq 1} A_n, \bigcap_{n \in \mathbb{R}, n \geq 1} A_n$, where $A_n = \{x \in \mathbb{R} : |x - 1| \geq \frac{1}{n}\}$.

PROBLEM 3.4: Let $A = \{a, b\}$, $B = \{1, 2, 3\}$.

(a) Compute $A \times B, \mathcal{P}(A \times B), \mathcal{P}(A) \times \mathcal{P}(B)$.

(b) What is the relationship between $\mathcal{P}(A \times B), \mathcal{P}(A) \times \mathcal{P}(B)$ (is one a subset of the other or not)? What is the relationship between $|\mathcal{P}(A \times B)|, |\mathcal{P}(A) \times \mathcal{P}(B)|$?

PROBLEM 3.5: Decide if each of the following is a function. In each case, decide also whether it is one-to-one, onto.

(a) $R1 = \{(x, y) \in [-3, 3] \times [-2, 2] : x^2/9 + y^2/4 = 1\}$.

(b) $R2 = \{(x, y) \in [-3, 3] \times [0, 2] : x^2/9 + y^2/4 = 1\}$.

(c) $R3 = \{(x, y) \in [-3, 0] \times [-2, 2] : x^2/9 + y^2/4 = 1\}$.

(d) $R4 = \{(x, y) \in [-3, 0] \times [0, 2] : x^2/9 + y^2/4 = 1\}$.

4. RELATIONS, EQUIVALENCE RELATIONS

Review the definitions of a relation R and R^{-1} (as well as $dom(R), codom(R), dom(R^{-1}), codom(R^{-1})$) as well as their basic properties (e.g. Theorem 7.3). Be able to come up with examples of relations with/without some given properties (reflexive, symmetric, transitive etc.)

Learn the definition of an equivalence relation, equivalence class (you should remember these definitions). Understand the correspondence between equivalence relations and partitions (Theorem 7.12). Study carefully an important example of equivalence relations $\equiv \pmod n$, operations $+_n, \times_n$ on equivalence classes of this relation (and why these are well-defined). Study Theorem 7.20.

PROBLEMS: Again, do the homework problems and worksheet (if any). The following are additional problems.

PROBLEM 4.1.

(a) Do problem 7.3.1.

(b) Do problem 7.3.14.

(c) Give an example of a relation that is both anti-symmetric and transitive, .

(d) A relation R is *asymmetric* if whenever $(x, y) \in R$, $(y, x) \notin R$. Give an example of a relation R that is both asymmetric and transitive. Is there an example of a relation R that is both asymmetric and antisymmetric? Explain.

PROBLEM 4.2.

(a) Do problem 7.5.2.

(b) Do problem 7.6.2.

(c) Do problems 7.6.3, 7.6.4.

PROBLEM 4.3. Prove or disprove: Let R_1, R_2 be two equivalence relations on a nonempty set X. Then $R_1 \cup R_2$ is an equivalence relation on X.

PROBLEM 4.4. Let E be a relation on $\mathbb{R} \times \mathbb{R}$ defined as: $(a, b)E(c, d)$ if $a + b = c + d$.

(a) Show that E is an equivalence relations.

(b) Let $+_E$ be defined on $\mathbb{R} \times \mathbb{R}/E$ as follows: $[(a, b)] +_E [(c, d)] = [(a + c, b + d)]$. Show hat $+_E$ is well-defined.