MATH 150 PRACTICE PROBLEMS FOR FINAL

1. Determine if the following are tautologies:

(a) $(R \to (S \vee Q)) \vee (R \vee (S \to Q))$

(b) $(R \leftrightarrow P) \vee (P \to \neg R)$.

2. The soundness theorem says that:

(a) If $\Gamma \vdash$ then $\Gamma \vDash \varphi$.

(b) If Γ is satisfiable (i.e. there is some model $\mathfrak{M} \vDash \Gamma$), then Γ is consistent.

Show that the two statements are equivalent.

3. The completeness theorem says that:

(a) If $\Gamma \vDash$ then $\Gamma \vdash \varphi$.

(b) If Γ is consistent, then Γ is satisfiable.

Show that the two statements are equivalent.

4. The compactness theorem says that:

(a) If $\Gamma \vDash$ then $\Gamma_0 \vdash \varphi$ for some finite $\Gamma_0 \Gamma$.

(b) If every finite subset of Γ is satisfiable, then so is Γ.

Show that the two statements are equivalent.

5. Consider the following extension of the language of rings.

- $\mathcal{L}_f = \{\dot{+}, \dot{\times}, \dot{0}, \dot{1}, \dot{<}, \dot{f}\}$ is the language of rings with an additional binary relation symbol $\dot{<}$ and a unary function symbol $\dot{f}$.

Consider the structure

$$\mathfrak{R} = (\mathbb{R}, +, \cdot, 0, 1, <, f)$$

where $\mathbb{R}$ is the set of all real numbers, and the interpretations of symbols $\dot{+}, \dot{\times}, \dot{0}, \dot{1}, \dot{<}$ in these structures are natural: $0, 1$ are numbers "zero" and "one", $+$ and $\cdot$ are usual addition and multiplication, and $<$ is the usual ordering of real numbers. Additionally, $\dot{f}$ is interpreted in $\mathfrak{R}$ as a unary function $f : \mathbb{R} \to \mathbb{R}$.

Express the following statements about numbers and function f in the structure $\mathfrak{R}$ as instructed below.

(a) Find an $\mathcal{L}$-formula $\varphi(u, v)$ which expresses:

"v is a local minimum of f at u".

(b) Find an $\mathcal{L}$-sentence τ which expresses:

"The set of arguments (i.e. points) at which f has local minimum is unbounded".

Now suppose $f(x) = e^x$ for all $x \in \mathbb{R}$; let $s : V \to \mathbb{R}$ be the evaluation of variables such that $s(x_{2k+1}) = 1$ for all $k \in \mathbb{N}$. Let t be the term

$$\text{“}\dot{\times}(\dot{f}(\dot{1}), \dot{+}(\dot{1}, \dot{f}(v_1)))\text{”}$$

and $\varphi(x)$ be the formula:

$$\exists v_2 (v_2 = \dot{f}(x)).$$

(c) Evaluate $t^{\mathfrak{R}}[s]$.

(d) Is t substitutable for x in φ? If so, determine whether $\mathfrak{R} \vDash \varphi(x/t)[s]$.

Explanations. Number b is a local minimum of a function f at a iff $f(a) = b$, and there is an open interval (x, y) containing a such that $f(z) \geq b$ for all $z \in (x, y)$. A set $X \subseteq \mathbb{R}$ is unbounded iff X has some element outside of any open interval (x, y) where $x, y \in \mathbb{R}$.

6. Consider a language $\mathcal{L}$ with a 2-ary predicate symbol $\dot{<}$. Let $\mathfrak{N} = (\mathbb{N}; <)$ be the structure of $\mathcal{L}$ consisting of the natural numbers with the usual ordering. Show that one cannot express the following statement in English

"There is no infinite descending chain."

by a sentence in the language $\mathcal{L}$. **Hint.** You may want to use the Compactness Theorem here. Think about what would happen if you could express the statement by a sentence τ in $\mathcal{L}$. Does $\mathfrak{N} \vDash \tau$? Can you find a model of $\mathcal{L}$ that satisfies τ?

7. Show that $\{\forall x(\alpha \to \beta), \exists x \alpha\} \vDash \exists x \beta$.

8. Let $\mathfrak{A} = (\mathbb{R}; +, \times)$ be an $\mathcal{L}$-structure, here $\mathcal{L}$'s nonlogical symbols are $\{\dot{+}, \dot{\times}\}$. Define the following sets in the structure $\mathcal{A}$.

(a) $\{0\}$.

(b) $\{1\}$.

(c) $\{3\}$.

(d) The interval $(0, \infty)$.

(e) $\{\langle r, s \rangle \mid r \leq s\}$ (here r, s are reals, of course).

9. Let $\mathfrak{A} = (\mathbb{N}; 0, 1, +, \times)$. Give a formula in the language of $\mathfrak{A}$ which defines the following. (Notice here that the language of $\mathfrak{A}$ only consists of the following non-logical symbols: $\dot{0}, \dot{1}, \dot{+}, \dot{\times}$).

(a) $\{2\}$.

(b) $\{n \mid n \text{ is even}\}$.

(c) $\{\langle m, n\rangle \mid m \text{ divides } n\}$.

(d) $\{n \mid n \text{ is a prime}\}$.

10. Assume that the language has a unary function symbol f. Find a sentence σ such that:

(a) for any model $\mathfrak{A}$, $\mathfrak{A} \models \sigma$ iff the universe of $\mathfrak{A}$ has at least two elements.

(b) for any model $\mathfrak{A}$, $\mathfrak{A} \models \sigma$ iff the universe of $\mathfrak{A}$ has exactly two elements.

(c) for any model $\mathfrak{A}$, $\mathfrak{A} \models \sigma$ iff $f^{\mathfrak{A}}$ is onto.

11. Consider the model ${}^*\mathfrak{R}$ discussed in class (and defined in Section 2.8). We also have standard structure $\mathfrak{R}$, where $|\mathfrak{R}| = \mathbb{R}$, $P_R^{\mathfrak{R}} = R, c_r^{\mathfrak{R}} = r$, $f_F^{\mathfrak{R}} = F$ for each relation symbol P_R, constant symbol c_r, and function symbol f_F. By the construction of ${}^*\mathfrak{R}$, $\mathfrak{R} \subset |{}^*\mathfrak{R}| =_{def} {}^*\mathbb{R}$. Let $<^* = P_<^{{}^*\mathfrak{R}}$.

(a) Show that for any $r, s \in \mathbb{R}$, there is some $t \in {}^*\mathbb{R}$ such that $r <^* t <^* s$.

(b) Show that there is $\epsilon \in {}^*\mathbb{R}$, $0 <^* \epsilon$ such that for positive $r \in \mathbb{R}$, $\epsilon <^* r$.

(c) Show that the set $\mathbb{R}$ is a bounded subset of ${}^*\mathbb{R}$. And there is no least upper bound for $\mathbb{R}$ in ${}^*\mathbb{R}$.